Masters Comprehensive Exam

Topology

January 13, 2003

1. Suppose that X and Y are topological spaces and $f : X \to Y$ is continuous. Prove or give a counter example:

   (a) If $A \subset X$ and $f(A)$ is open in Y, then A is open in X.

   (b) $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.

2. Let X and Y be topological spaces, let $X \times Y$ have the product topology and let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projections. Suppose that $C \subset X \times Y$ is closed and that $\pi_X(C)$ and $\pi_Y(C)$ are compact. Prove that C is compact.

3. Let (X, d) be a connected metric space. If $a, b \in X$ are such that $d(a, b) = r \geq 0$, prove that for each s, $0 \leq s \leq r$, there is a point $x_s \in X$ such that $d(a, x_s) = s$.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and also has the property that if $A \subset \mathbb{R}$ is unbounded then so is $f(A)$. Prove that f is a closed map (that is, if C is closed then $f(C)$ is also closed).