Topology

Master's Exam

January 7, 2008

Instructions: Work at most one problem on a side of paper.

(1) Define a topology on the plane $\mathbb{R}^2$ as follows: at each point p in the plane, the basic open neighborhoods are the sets $\{p\} \cup D$ where D is a disc about p with a finite number of straight lines removed. *Assume without proof that this really does define a unique topology.*

  (a) Compare this topology with the usual (Euclidean) topology on the plane.

  (b) If "finite" is replaced by "countable" in the definition of this topological space, is it still the same topological space?

(2) Suppose X and Y are topological spaces. Prove that $f : X \to Y$ is continuous if and only if for all sets $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$. Please do not cite a theorem here. We are looking for an elementary proof.

(3) Suppose that $f : X \to Y$ is a continuous map from the metric space (X, d_X) to the metric space (Y, d_Y). Suppose further that

  (a) for all $x_0 \in X$ and real numbers $r \geq 0$, the closed ball

$$\overline{B}(x_0, r) = \{x \in X \,:\, \mathsf{d}_X(x, y) \leq r\}$$

  is compact; and,

  (b) if $Z \subset X$ is unbounded[1], then $f(Z) \subset Y$ is unbounded.

  Prove that if K is a compact subset of Y, then $f^{-1}(K)$ is a compact subset of X.

(4) Show that the topological space X is connected if and only if the boundary of every proper and nonempty subset A of X is not empty[2].

[1] means Z is not contained in any ball.

[2] Recall that the boundary of A is the intersection of the closure of A with the closure of $X \backslash A$.