Masters Comprehensive Exam - Topology

January, 2010

CHOOSE THREE

In the following, $\mathbb{R}^n$ denotes n-dimensional Euclidean space with the usual topology.

(1) For each $n \in \mathbb{N}$ let $\mathbf{X}_n := \{(1/n, y) : 0 \le y \le 1\}$, let $\mathbf{X}_\infty := \{(0, y) : 0 \le y \le 1\}$, and let $\mathbf{Y}_0 := \{(x, 0) : 0 \le x \le 1\}$. Define $\mathbf{C}$ to be the subspace of $\mathbb{R}^2$ given by $\mathbf{C} := \mathbf{X}_\infty \cup (\cup_{n\in\mathbb{N}} \mathbf{X}_n) \cup \mathbf{Y}_0$. Prove that there is no continuous map from the unit interval $[0, 1]$ onto $\mathbf{C}$.

(2) Let $\sim$ denote the equivalence relation on $\mathbb{R}^n \setminus \{\mathbf{o}\}$ defined by $\mathbf{x} \sim \mathbf{y}$ iff there is a $t \in \mathbb{R}$ such that $\mathbf{x} = t\mathbf{y}$. Let $\mathbb{P}^n$ be the quotient space $\mathbb{P}^n := (\mathbb{R}^n \setminus \{\mathbf{o}\})/\sim$.
(a) Prove that $\mathbb{P}^n$ is connected.
(b) Prove that $\mathbb{P}^n$ is compact.

(3) A continuous map $\alpha : [a, b] \to \mathbb{R}^n$ is a piece-wise linear arc from $\mathbf{x}$ to $\mathbf{y}$ provided there is an $n \in \mathbb{N}$, there are t_i in $[a, b]$ with $a = t_0 < t_1 < \cdots < t_n = b$, and $\mathbf{x}_i \in \mathbb{R}^n$, such that $\mathbf{x}_0 = \mathbf{x}, \mathbf{x}_n = \mathbf{y}$, and

$$\alpha(t) = (1 - \frac{(t - t_i)}{(t_{i+1} - t_i)})\mathbf{x}_i + \frac{(t - t_i)}{(t_{i+1} - t_i)}\mathbf{x}_{i+1},$$

for $t_i \le t \le t_{i+1}$ and $i = 0, \ldots, n - 1$.
Prove that if $\mathbf{U}$ is a connected open set in $\mathbb{R}^n$, then $\mathbf{U}$ is piece-wise linearly arc-connected (that is, for each $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{U}$, there is a piece-wise linear arc from $\mathbf{x}$ to $\mathbf{y}$ with image in $\mathbf{U}$).

(4) (TRUE or FALSE) Decide for each statement below if it is true or false and support your conjecture by either a proof or a counterexample.

a.: Each contraction is uniformly continuous. (A map $f : X \to X$, where (X, d) is a metric space is a contraction if there is $\alpha < 1$ such that $d(f(x), f(y)) \le \alpha d(x, y)$ for all $x, y \in X$ with $x \ne y$.)

b.: Every closed set is compact.

c.: If $g : X \to Y$ is an inclusion, then $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ is an injection.

d.: If Y is contractible, then for all topological spaces X, $[X, Y]$ has a single element. (Y is said to be contractible if the identity map $i_Y : Y \to Y$ is null-homotopic.)