Complex Analysis Ph. D. Comprehensive Exam
August 2016

Solve 4 of the following 5 problems.

1. Let f be analytic with a zero of order $N \geq 1$ at z_0. Show that there exists an analytic N-th root of f in a neighborhood of z_0, i.e., an analytic function g such that $g(z)^N = f(z)$ in some disk about z_0.

2. Find a conformal map from the half-strip $D = \{z = x + iy \mid 0 < x < 1,\ y > 0\}$ onto the upper half-plane $H = \{z = x + iy \mid y > 0\}$. (Hint: Trigonometric functions might help.)

3. Let f be an entire function with $\limsup\limits_{z\to\infty} \frac{\log|f(z)|}{\log|z|} < \infty$. Show that f is a polynomial. (Hint: First show that the assumption implies that there exist $N > 0$ and $R > 0$ such that $|f(z)| \leq |z|^N$ for $|z| \geq R$.)

4. Find
$$\int_0^\infty \frac{dx}{x^6 + 1}$$

5. Does there exists a function f which is analytic in the unit disk and satisfies $f(1/n) = f(-1/n) = 1/n^3$ for all $n = 2, 3, \ldots$? Justify your answer.