PhD Dynamics Exam *(Aug 2016)*

NAME:

Pick, Circle, and Solve 5 problems. Good luck!

1. Consider a continuous map $f : X \to X$ of a compact metric space. Show that
a) for any $x_0 \in X$, the omega limit set $\omega(x_0)$ (which consist of all limits $\lim_{k\to\infty} f^{n_k}(x_0)$ where $n_k \to \infty$) is contained in $Y := \bigcap_{n\geq 0} f^n(X)$;
b) there exists $x_0 \in X$ such that $x_0 \in \omega(x_0)$.

2. Prove that, for any triple d_0, d_1, d_2 of decimal digits, there is $n \in \mathbb{N}$ such that 3^n has decimal expansion that starts with the sequence, that is $3^n = d_0 d_1 d_2 \ldots$. (You can use without proof that $\log_{10} 3$ is irrational.)

3. For a continuous map $f : X \to X$ of a compact metric space, show that f is topologically transitive (i.e., has a point with dense orbit) if $\bigcup_{n\in\mathbb{N}} f^n(U)$ is dense in X for any non-empty open $U \subset X$.

4. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be induced by some integer matrix A, i.e., $f(x) = Ax$, with the arithmetic modulo one. ($\mathbb{T}^2$ is the usual flat torus $\mathbb{R}^2/\mathbb{Z}^2$.) Prove that every point $x \in \mathbb{T}^2$ with rational coordinates is pre-periodic (i.e., $f^n(x)$ is periodic for some $n \geq 0$). Additionally, argue that if all such points are actually periodic then A is invertible over the integers (i.e., $\det(A) = \pm 1$).

5. Consider continuous maps of compact metric spaces $f : X \to X$ and $g : Y \to Y$ where g is a factor of f (i.e., $\phi \circ f = g \circ \phi$ for some continuous surjective $\phi : X \to Y$). Prove that $h_{\text{top}}(g) \leq h_{\text{top}}(f)$. Give an example of f, g, and the factor map ϕ where $h_{\text{top}}(f) = h_{\text{top}}(g)$ even though f and g are not topologically conjugate.

6. Give an example of a subshift X that is not a subshift of finite type (SFT). (This includes showing that your X is not an SFT, i.e., there is not a finite forbidden set defining X.)

7. Let ϕ be the substitution $1 \mapsto 121$, $2 \mapsto 1$ and $w = w_1 w_2 w_3 \ldots$ be an infinite fixed word, i.e., $\phi(w) = w$. Compute the *frequency* of the symbol 1 in w, as given by

$$\nu(1) := \lim_{n\to\infty} \frac{1}{n} \# \{k \in \{1, \ldots, n\} : \quad w_k = 1\}.$$

You can take for granted that the limit exists.