Departments of Mathematics
Montana State University

Fall 2016

Real Analysis Comprehensive Exam

9:00 am - 1:00 pm, Monday, August 22nd, 2016

Solve 4 of the following 5 problems. Clearly mark the solutions to be graded.

1. Let $E \subseteq \mathbb{R}$ be a set.
   (a) Define the outer measure $m^*(E)$.
   (b) Show that for all $\epsilon > 0$, there is an open subset U of $\mathbb{R}$ that contains E and satisfies
   $$m^*(U) \leq m^*(E) + \epsilon.$$

2. Consider the collection $\mathcal{M}$ of subsets of $\mathbb{R}$ defined by
   $$\mathcal{M} := \{A \subseteq \mathbb{R} : \ A \text{ is countable or } \mathbb{R} \backslash A \text{ is countable}\}.$$
   (a) Show that $\mathcal{M}$ is a σ-algebra.
   (b) Give (with proof) an example of a function that is measurable with respect to $\mathcal{M}$ and an example of a function that is not measurable with respect to $\mathcal{M}$.

3. Let X be a set, Σ a σ-algebra on X, and μ a measure on Σ. Suppose that $\{f_n : X \to \mathbb{R}\}_{n \in \mathbb{N}}$ is a sequence of measurable functions that converges in measure to a measurable function $f : X \to \mathbb{R}$. Show that
   $$\int_X f \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.$$

4. Let $p > 1$ and $q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in \mathrm{L}_p(\mathbb{R})$. For $g \in \mathrm{L}_q(\mathbb{R})$, define
   $$A_f(g) = \int_X fg \, d\mu.$$
   (a) Show that A_f defines a bounded linear functional on $\mathrm{L}_q(\mathbb{R})$.
   (b) State the Riesz Representation Theorem relevant to this situation.

5. Let $n \geq 2$ be an integer, and let $P = \{p_1, \ldots, p_n\}$ be a finite collection of distinct real numbers. Consider the measure on the Lebesgue σ-algebra in $\mathbb{R}$ by
   $$\mu(E) = \frac{1}{n} \operatorname{card}(E \cap P).$$
   (a) Give an example of a measure ν on the Lebesgue σ-algebra that satisfies
   - ν is not a constant multiple of μ,
   - $\nu(\mathbb{R}) = 1$, and
   - ν is absolutely continuous with respect to μ

   Prove only the absolute continuity.
   (b) For your example, find (with proof) the Radon-Nikodym derivative of ν with respect to μ.