Applied Mathematics Comprehensive Exam

August 2015

Instructions: Answer 3 of the problems from **Part A** and answer 3 problems from **Part B**. Indicate clearly which questions you wish graded.

Part A

A.1 (a) Find the pseudo inverse $A^\dagger$ of the matrix:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$$

(b) If $\mathbf{b}^T = (1,1,1,1)$ find the Least Squares solution of $A\mathbf{x} = \mathbf{b}$.

(c) Clearly state conditions on $\mathbf{y}$ so that $A\mathbf{x} = \mathbf{y}$ has a solution.

A.2 Define

$$\mathbf{F}(\mathbf{x}, \epsilon) = \begin{pmatrix} f_1(x,y,\epsilon) \\ f_2(x,y,\epsilon) \end{pmatrix} = \begin{pmatrix} x - y - 4\epsilon x^3 \\ x - y - \epsilon y \end{pmatrix}$$

A regular expansion for the (nonzero) root $\mathbf{x}$ of $\mathbf{F}(\mathbf{x}, \epsilon) = 0$ is:

$$\begin{aligned} x &= \alpha + \epsilon x_1 + O(\epsilon^2) \\ y &= \alpha + \epsilon y_1 + O(\epsilon^2) \end{aligned}$$

for some α. Use the Fredholm alternative to find α.

A.3 Let $H(x)$ be the Heaviside function. Prove

$$\lim_{\epsilon \to 0^+} \frac{H(x+\epsilon) - H(x)}{\epsilon} = \delta(x)$$

in the distribution sense where δ is the delta distribution.

A.4 Let $Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u$ be a linear second order differential operator on

$$D(L) = \{u \in C^2[1,2] : u(1) = 0, u(2) = 0\}$$

(a) Compute the adjoint of L (and domain) and then state what distributional equation the associated Green's function $g(x,\zeta)$ must satisfy.

(b) Explicitly compute the Green's function $g(x,\zeta)$ for the self adjoint operator

$$Lu = xu'' + u'$$

Part B

B.1 A functional J on admissible set $\mathcal{A}$ are defined:

$$\begin{aligned} J(y) &= y(1)^2 + \int_0^1 L(x, y, y')dx \\ \mathcal{A} &= \{y \in C^2[0,1] : y(0) = 3\} \end{aligned}$$

where $L = xy + (y')^2$. Let $\bar{y}(x)$ be the extrema of J over $\mathcal{A}$.

(a) Derive the Euler Lagrange equation and natural boundary condition $\bar{y}(x)$ must satisfy.

(b) Find the extrema $\bar{y}(x)$.

B.2 Curves on a cone are described parametrically in polar coordinates by

$$\begin{aligned} X(t) &= r(\theta)\cos\theta \\ Y(t) &= r(\theta)\sin\theta \\ Z(t) &= r(\theta) \end{aligned}$$

where $\theta = \theta(t)$ is a function of the parameter t. Geodesics on the cone extremize

$$J(r) = \int_{t_1}^{t_2} \| \dot{\mathbf{X}} \| \, dt = \int_{\theta_1}^{\theta_2} L(r, r')d\theta \quad , \quad \mathbf{X} = (X, Y, Z)$$

(a) Derive the Lagrangian $L(r, r')$.

(b) Find a first integral for the Euler-Lagrange equation.
Do not attempt to solve it.

(c) Is J a strictly convex?

B.3 Find two term expansions (in $\epsilon \ll 1$) for all singular roots of

$$\epsilon x^3 - \frac{x}{x+1} = 0$$

B.4 Find the inner and outer solution approximations of the interior layer solution of

$$\begin{aligned} \epsilon y'' + xy' + xy &= 0 \quad , \quad x \in (-1, 1) \\ y(-1) &= 2 \\ y(1) &= -2 \end{aligned}$$

You may need the error function $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $erf(x) \to 1$ as $x \to \infty$.