Functional Analysis Qualifying Examination, August 2015

Name:

Try to completely answer 5 of the 8 problems. State any standard theorems you use. Good luck!

1. Consider

$$\ell_2 = \left\{ x = \{x_n\}_{n=1}^{\infty} : x_n \in \mathbb{R} \text{ and } \|x\|_{\ell_2} = \sqrt{\sum_{n=1}^{\infty} |x_n|^2} < \infty \right\}.$$

Show that $\|\cdot\|_{\ell_2}$ is a norm.

2.a. Let X be a normed linear space, C a closed convex subset of X such that $\lambda x \in C$ whenever $x \in C$, and $|\lambda| \leq 1$, $w \in X \setminus C$, and $B \in X$ an open ball around the origin such that $B + w$ does not intersect C. Show that the Minkowski functional $p : X \to [0, \infty)$ defined by

$$p(x) = \inf\{t > 0 : x/t \in C + B\}$$

is a seminorm.
b. Use this to show that there exists an $f \in X^*$ such that $|f(x)| \leq 1$ for all $x \in C$ and $f(w) > 1$.

3.a. Prove $C([0,1])$, the space of continuous functions on the closed interval, is not complete in the L_1 metric: $\rho(f,g) = \int_0^1 |f(x) - g(x)| dx$. Note that this space is complete under $\|\cdot\|_{\ell_\infty}$.
b. What can be said if we have another norm on the space, call it $\|\cdot\|$, and a constant C such that $\|x\| \leq C\|x\|_{\ell_\infty}$ for all $x \in C([0,1])$ but the norms are known not to be equivalent?

4. Show that every orthonormal sequence in a Hilbert space converges weakly to zero.

5.a. State the Spectral Theorem for Compact Operators.
b. State the Fredholm Alternative and prove it by using part a).

6. Consider the complex Hilbert space $\ell_2(\mathbb{Z})$ of sequences $x_n \in \mathbb{C}$, $n \in \mathbb{Z}$ such that

$$\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty.$$

Define the shift operator $S : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ by $S((x_n)) = (x_{n+1})$. Find the point spectrum and the continuous spectrum of S.

7.a. Show that the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{\pi}{2} + x - \arctan x$$

has no fixed points in $\mathbb{R}$.
b. Show that $|T(x) - T(y)| < |x - y|$ for all distinct pairs of $x, y \in \mathbb{R}$. Explain why this does not contradict the contraction mapping theorem.

8.a Let (x_k) be a linearly independent system in a Hilbert space X. Use this to produce an orthonormal system in X. What can be said about the span of your new orthonormal system?

b. Outline in very broad terms how we know Sturm-Liouville problems produce orthonormal bases.