ALGEBRA
PHD COMPREHENSIVE EXAMINATION

Instructions. You are welcome to do or use any subproblem of a problem without doing the entire problem. Partial credit will be generously given to solutions demonstrating the key ideas, such as in outline-form, even if not implemented entirely/perfectly/rigorously.

(1) (a) For each positive number $d > 0$, prove that the polynomial $x^d - 2 \in \mathbb{Q}[x]$ is irreducible.

(b) Let $d > 0$ be a positive number. Use the above point to conclude that the field extension $\mathbb{Q}[2^{\frac{1}{d}}]$ of $\mathbb{Q}$ has degree d.

(c) Consider the algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers. The canonical injection $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$ between fields endows the underlying abelian group of $\overline{\mathbb{Q}}$ with the structure of a $\mathbb{Q}$-vector space. Use the previous point to prove that, as a $\mathbb{Q}$-vector space, $\overline{\mathbb{Q}}$ is infinite-dimensional.

(d) For $p(x) \in \mathbb{Q}[x]$ a polynomial with rational coefficients, denote by $\mathsf{Split}_{\mathbb{Q}}(p)$ a minimal splitting field for $p(x)$.** Use the previous point to conclude that there does **not** exist a polynomial $p(x) \in \mathbb{Q}[x]$ for which $\mathsf{Split}_{\mathbb{Q}}(p)$ is algebraically closed.

(**More precisely, $\mathsf{Split}_{\mathbb{Q}}(p)$ is a minimal field extension of $\mathbb{Q}$ for which there are degree 1 polynomials $p_1(x), \ldots, p_d(x) \in \mathsf{Split}_{\mathbb{Q}}(p)[x]$ and an equality $p(x) = \prod_{1 \leq i \leq d} p_i(x)$ in $\mathsf{Split}_{\mathbb{Q}}(p)[x]$.)

(2) Let G be a finite abelian group with at least 2 elements. Prove, for find a counter-example to, each of the following assertion concerning group-rings associated to G.

(a) The following data exists.

- There is a field $\mathbb{F}$ whose characteristic is prime to the order $|G|$.
- There is a number $s \geq 0$.
- There are polynomials

$$q_1(x), q_2(x), \ldots, q_s(x) \in \mathbb{F}[x] .$$

- Consider a minimal field extension $\mathsf{Split}_{\mathbb{F}}(p)$ of $\mathbb{F}$ in which, for each $1 \leq i \leq s$, the polynomial $q_i(x)$ has all of its roots. There is an isomorphism between $\mathbb{F}$-algebras:

$$\mathsf{Split}_{\mathbb{F}}(p) \cong \mathbb{F}[G] .$$

Hint: Construct a surjective ring homomorphism $\mathbb{F}[G] \to \mathbb{F}$. Use this to conclude that $\mathbb{F}[G]$ is not a field.

(b) The following data exists.

- There is a number $r \geq 0$.
- There are polynomials

$$p_1(x), p_2(x), \ldots, p_r(x) \in \mathbb{C}[x] .$$

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- Consider the ideal $I = (p_1(x_1)) + (p_2(x_2)) + \cdots + (p_r(x_r)) \subset \mathbb{C}[x_1, \ldots, x_r]$. There is an isomorphism between $\mathbb{C}$-algebras:
$$\mathbb{C}[x_1, \ldots, x_r]/I \cong \mathbb{C}[G] .$$
Hint: First establish the case that G is cyclic: take $r = 1$ and the roots of $p_1(x)$ to be roots of unity. Use a property of how Group Rings interact with products of groups to conclude the case that G is a product of cyclic groups. Employ the Fundamental Theorem of Finite Abelian Groups.

(3) Consider the canonical ring homomorphism $\mathbb{R}[x] \to \mathbb{R}[x,y]$, regarding a polynomial in x as a polynomial in x and y. The canonical ring homomorphisms
$$\mathbb{R} \longrightarrow \mathbb{R}[x] \longrightarrow \mathbb{R}[x,y] \longrightarrow \mathbb{R}[x,y]/(y^2)$$
endow any $\mathbb{R}[x,y]$-module, such as $\mathbb{R}[x,y]/(y^2)$, with the structure of an $\mathbb{R}[x]$-module, which in turn forgets to the structure of an $\mathbb{R}$-vector space. Let M be a non-zero $\mathbb{R}[x,y]$-module that is finite-dimensional as an $\mathbb{R}$-vector space.

(a) Consider the abelian group
$$\mathrm{Hom}_{\mathbb{R}[x,y]}\left(\mathbb{R}[x,y]/(y^2), M\right) ,$$
with its natural $\mathbb{R}$-vector space structure. Construct an injection from this $\mathbb{R}$-vector space into the underlying $\mathbb{R}$-vector space of M. Conclude that this $\mathbb{R}$-vector space is finite-dimensional.

(b) Consider the abelian group
$$\mathrm{Hom}_{\mathbb{R}[x]}\left(\mathbb{R}[x,y]/(y^2), M\right) ,$$
with its natural $\mathbb{R}$-vector space structure. Prove that this $\mathbb{R}$-vector space is finite-dimensional. Identify its dimension in terms of the dimension $\dim_{\mathbb{R}}(M)$.

(c) Consider the abelian group
$$\mathrm{Hom}_{\mathbb{R}}\left(\mathbb{R}[x,y]/(y^2), M\right) ,$$
with its natural $\mathbb{R}$-module structure. Prove that this $\mathbb{R}$-vector space is infinite-dimensional.

(4) Let R be a commutative ring (with $1 \neq 0$). Let $M \subset R$ be a (unital) maximal ideal.

(a) Let $u \in R$ be a unit, and let $v \in R$ be a nilpotent element. Prove that the element $u + v \in R$ is a unit.

(b) For $i > 0$, use the above point to prove that R/M^i is a local ring.
(Recall that a commutative ring is a *local ring* if there is a unique maximal ideal in it.)
Hint: Use that each element of R/M^i can be represented by an element of R of the form $u + m_1 + m_2 + \cdots + m_i$ where $u \notin M$ and each $m_i \in M^i$.

(c) Prove that the limit ring $\widehat{R_M} := \lim\limits_{i>0} R/M^i$ is local ring.
Hint: For each $i > 0$, consider the canonical ring homomorphism $\widehat{R_M} \to R/M^i$. Show that an element in $\widehat{R_M}$ is a unit if and only if, for each $i > 0$, its value in R/M^i is a unit.

(d) Use the above point to prove that the ring $\mathbb{R}[[x]]$ of formal power series is a local ring.

(e) Use the twice-above point to prove that, for p a prime number, the ring $\mathbb{Z}_p$ of p-adic integers is a local ring.

(5) Let $\rho : G \to H$ and $\iota : H \to G$ be homomorphisms between groups. Suppose the composition $\rho \circ \iota = \mathrm{id}_H$ is the identity map on H.

(a) Construct a homomorphism $\varphi\colon H \to \mathsf{Aut}_{\mathsf{Groups}}\big(\mathsf{Ker}(\rho)\big)$ to the group of automorphisms of the group $\mathsf{Ker}(\rho)$, which is the kernel of ρ.

(b) With respect to the above homomorphism φ, construct an isomorphism between groups,

$$G \cong H \underset{\varphi}{\ltimes} \mathsf{Ker}(\rho) ,$$

where the righthand group is the semi-direct product classified by φ. (For full credit, you must construct a homomorphism, and show that it is an isomorphism.)

(c) Consider the group $\mathsf{GL}(\mathbb{R}^n)$ whose underlying set consists of those $n \times n$ matrices with real entries whose determinant is not zero, and whose group structure is matrix multiplication. Consider the subgroup

$$\mathsf{SL}(\mathbb{R}^n) := \big\{A \in \mathsf{GL}(\mathbb{R}^n) \mid \det(A) = 1\big\} \subset \mathsf{GL}(\mathbb{R}^n) .$$

Use the above points to deduce the existence of an isomorphism between groups:

$$\mathsf{GL}(\mathbb{R}^n) \cong \mathbb{R}^\times \ltimes \mathsf{SL}(\mathbb{R}^n) .$$

(d) Suppose $n > 1$. Consider the subgroup

$$\mathsf{O}(n) := \big\{A \in \mathsf{GL}(\mathbb{R}^n) \mid A^T = A^{-1}\big\} \subset \mathsf{GL}(\mathbb{R}^n)$$

of orthogonal $n \times n$ matrices. Prove that $\mathsf{O}(n)$ is **not** a factor in a semi-direct product factorization of $\mathsf{GL}(\mathbb{R}^n)$; more precisely, prove that the following data does **not** simultaneously exist:

- A group H.
- A homomorphism $\varphi\colon H \to \mathsf{Aut}_{\mathsf{Groups}}\big(\mathsf{O}(n)\big)$.
- An isomorphism between groups:

$$\mathsf{GL}(\mathbb{R}^n) \cong H \underset{\varphi}{\ltimes} \mathsf{O}(n) .$$