Applied Mathematics Ph. D. Comprehensive Exam

August 2017

Instructions: Answer 3 of the problems from **Part A** and 3 problems from **Part B**. Indicate clearly which questions you wish graded.

Part A

A.1 In the following we consider the system $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & \alpha \end{bmatrix} \quad , \quad b = \begin{pmatrix} 1 \\ 2 \\ \beta \end{pmatrix}$$

(a) Write out the Normal equations for the least squares solution of $Ax = b$.
(b) Find all values of α is for which the least squares solution is unique.
(c) When $\alpha = 1$ for what β does $Ax = b$ have a solution?

A.2 The following operator is self adjoint on $D(K) = L^2[0, 1]$

$$Ku \equiv \int_0^1 xy\ u(y)\ dy$$

(a) Find the nonzero eigenvalue λ_1 of K and an associated eigenfunction.
(b) Consider the perturbed nonlinear integral equation

$$(K - \lambda_1 I)u = \epsilon(u^2 - 2) \quad , \quad 0 < \epsilon \ll 1$$

with expansion $u(x) = u_0(x) + \epsilon u_1(x) + O(\epsilon^2)$. Use the Fredholm alternative to show the $O(\epsilon)$ problem for u_1 has a solution only for two $u_0 \in N(K - \lambda_1 I)$.

A.3 Define the operator $Lu \equiv u''$ with domain

$$D(L) = \{u \in C^2[0, 1] : u'(0) = 0, u(1) = 0\}$$

(a) Find the Green's function $g(x, t)$ for the problem $Lu = f$ such that

$$u(x) = \int_0^1 g(x, t) f(t)\ dt$$

(b) Use the eigenfunctions $\phi_n(x)$ of L to find a Fourier series representation of the Green's function in part (a).

A.4 Prove the Prinicipal Value distribution $t(x) = Pv\left(\frac{1}{x}\right)$ is well defined on the set D of smooth test functions with compact support. Then compute its distributional derivative $t'(x)$ over the subset of test functions ϕ in D having $\phi(0) = 0$.

Part B

B.1 Find the extrema of the functional

$$J(y) = \int_0^9 \sqrt{x + (y')^2}\, dx$$

over the admissible set

$$A = \{y \in C^2[0,9] : y(0) = 0, y(9) = 9\}$$

B.2 Define the functional

$$J(u) \equiv \int_\Omega L(x, u, \nabla u)\, dx \quad , \quad \Omega \subset \mathbb{R}^3$$

where Ω is a simply connected domain with smooth boundary $\partial\Omega$ and $u(x)$ is in the admissible set

$$A = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 4\}$$

Using the Divergence Theorem and the identity

$$\nabla \cdot (a\vec{b}) = \nabla a \cdot \vec{b} + a\nabla \cdot \vec{b}$$

to derive the Euler-Lagrange equations for the $u(x)$ that extremizes $J(u)$ over A.

B.3 Find two term expansions (in ϵ) for all real roots of

$$\epsilon x^3 - x + 1 = 0 \quad , \quad 0 < \epsilon \ll 1$$

B.4 Find the uniformly valid approximation for the boundary value problem:

$$\epsilon y'' + \frac{1}{1+x} y' + \epsilon y = 0 \quad , \quad y(0) = 0 \quad , \quad y(1) = 1$$

You may the only layer occurs at the boundary $x = 0$.