Complex Analysis Ph. D. Comprehensive Exam
August 2017

Pick, circle, and solve 4 of the following 5 problems.

1. Which of these analytic functions has a primitive (i.e., can be written as the derivative of an analytic function) in the indicated domain D? Justify your answers.

   (a) $f(z) = e^{-z^2}$ in $D = \mathbb{C}$.

   (b) $g(z) = e^{1/z}$ in $D = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

   (c) $h(z) = e^{1/z^2}$ in $D = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

   (Hint: Explicit integration is not the way to go in either of these.)

2. Let f be a conformal map from the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ onto some domain $D \subset \mathbb{C}$. Show that $D = \mathbb{C} \setminus \{w_0\}$ for some $w_0 \in \mathbb{C}$. (Hint: What type of isolated singularities can f have at 0 and at ∞?)

3. Let us say that a domain $D \subseteq \mathbb{C}$ is a *circle domain* if it is a bounded domain whose boundary components are disjoint circles. I.e., there exists points $z_0, \ldots, z_n \in \mathbb{C}$ and radii $r_0, \ldots, r_n > 0$ such that the closed disks $\overline{D_k} = \overline{D_{z_k}(r_k)}$ for $k = 1, \ldots, n$ are mutually disjoint subsets of $D_0 = D_{z_0}(r_0)$ with $D = D_0 \setminus \bigcup_{k=1}^{n} \overline{D_k}$.

   (a) Show that every analytic function $f : \overline{D} \to \mathbb{C}$ has a decomposition $f = f_0 + \ldots + f_n$ where f_0 is analytic in D_0, and each f_k for $k = 1, \ldots, n$ is analytic in $\Delta_k = \hat{\mathbb{C}} \setminus \overline{D_k}$ (where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e., $f_1, \ldots, f_n$ are analytic at ∞, too.)

   (b) Show that this decomposition is unique up to constants.

   (Hint: Mimic the proof of the Laurent decomposition. Note that it is assumed for simplicity that f is analytic in the closure of D, i.e., in a domain containing $\overline{D} = D \cup \partial D$.)

4. Let n be a positive integer and consider the equation $\cos z = 3z^n$.

   (a) Find the number of solutions (counted with multiplicity) in the unit disk.

   (b) Show that these are all solutions in the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$.

5. Let $f : \mathbb{D} \to \mathbb{D}$ be analytic with $f(1/2) = f(-1/2) = 0$. Show that

$$|f(z)| \leq \left| \frac{4z^2 - 1}{4 - z^2} \right|$$

   (Hint: Show first that the map $g(z) = (4z^2 - 1)/(4 - z^2)$ maps the unit disk to itself and fixes the unit circle. Then consider $h(z) = f(z)/g(z)$.)