Ph.D. Comprehensive Exam: Numerical Analysis

August 2017

1. The Lax-Friedrichs scheme for solving the advection equation $u_t + au_x = 0$ with constant velocity a is given by

$$U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) - \frac{a\Delta t}{2\Delta x}(U_{j+1}^n - U_{j-1}^n)$$

   (a) Calculate the leading terms of the truncation error for this method, assuming that $\nu = a\Delta t/\Delta x$ is a constant.

   (b) State the definition of CFL condition. For what values of ν does the Lax-Friedrichs scheme satisfy the CFL condition?

   (c) Find the amplification factor $\lambda(k)$ in the Fourier analysis of stability for the Lax-Friedrichs scheme. For what values of ν is the scheme stable?

2. (a) State the definition of consistency and convergence of a finite difference scheme

$$B_1 U^{n+1} = B_0 U^n + F^n$$

   where B_1, B_0 are linear difference operators, F^n is a known vector, and U^n is the numerical solution at time t_n.

   (b) State the Lax Equivalence Theorem.

3. Develop the weak form and the finite element model of the following differential equation over an element $\Omega_e = (x_a, x_b)$:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) - b\frac{du}{dx} = f, \quad \text{for} \quad x_a < x < x_b$$

   where a, b, f are known functions of x, and u is the dependent variable. Since no boundary condition is specified at the element level, one must identify the secondary variable(s) at the two ends of the element by some symbols (such as Q_1^e, Q_2^e) in developing the weak form. To develop the finite element model, assume approximation of the form $u_h(x) = \sum_{j=1}^n u_j^e \psi_j^e(x)$, substitute this expression for the primary variable and ψ_i^e for the weight function into the weak form, and calculate all coefficients of the model (K_{ij}^e, f_i^e) in terms of the problem data and ψ_i^e. Is K^e symmetric?

4. Solve the Poisson equation $-\nabla^2 u = 1$ on the unit square domain with boundary conditions given in the figure with one linear rectangular element. Show the following steps: weak formulation, element equation (assembly is not necessary since there is only one element), condensed equation resulted from imposing boundary conditions. The final answer should be the value of the approximate solution u_h at points $(1,1)$ and $(0,1)$.