Real Analysis Comprehensive PhD Exam
(Aug 2017)

NAME:

Pick, Circle, and Solve 4 problems. Good luck!

1. Suppose that μ and ν are finite measures (on the same σ-algebra $\mathcal{X}$) and that $\nu << \mu$ (i.e. $\mu(E) = 0 \implies \nu(E) = 0$ for all $E \in \mathcal{X}$). Show that, for any $\epsilon > 0$, there is $\delta > 0$ such that $\mu(E) < \delta \implies \nu(E) < \epsilon$ for all $E \in \mathcal{X}$.

2. Suppose that $\mathcal{A}$ is a collection of subsets of a set X that is closed under countable unions and finite intersections (i.e. $A_n \in \mathcal{A} \implies \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ and $A_1, A_2 \in \mathcal{A} \implies A_1 \cap A_2 \in \mathcal{A}$). A function $f : X \to \mathbb{R}$ is **$\mathcal{A}$-measurable** iff the sets $\{x \in X : \ f(x) > \alpha\}$ belong to $\mathcal{A}$ for all $\alpha \in \mathbb{R}$. Prove that, if f and g are $\mathcal{A}$-measurable, then $f + g$ is $\mathcal{A}$-measurable. (Please, make sure to carefully justify all steps.)

3. Show that any bounded linear functional $\xi : L^1(\mathbb{R}) \to \mathbb{R}$ can be decomposed, $\xi = \xi_1 + \xi_2$, into two bounded linear functionals $\xi_1, \xi_2 : L^1(\mathbb{R}) \to \mathbb{R}$ where ξ_1 is supported on $[0, \infty)$ and ξ_2 is supported on $(-\infty, 0]$. (A functional $\xi : L^1(\mathbb{R}) \to \mathbb{R}$ is **supported** on a segment J iff $\xi(f) = 0$ for any $f \in L^1(\mathbb{R})$ that a.e. vanishes on J.)

4. Without leaning on Lebesgue Density Theorem, prove that if the Lebesgue measure $\lambda(A)$ of a subset $A \subset [0, 1]$ is positive then there is a segment $J \subset \mathbb{R}$ such that $\lambda(A \cap J) > \frac{1}{2}\lambda(J)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable and essentially bounded. Define

$$F(x) := \int_{-\infty}^{\infty} f(x - t) f(x + t)\, dt.$$

(i) Show that F is a well defined element of $L^1(\mathbb{R})$ and that $\|F\|_1 \leq \frac{1}{2}\|f\|_1^2$.
(ii) Show that F is continuous.