PHD COMPREHENSIVE EXAM: TOPOLOGY

This exam lasts for 4 hours. You must complete both Part A and Part B of the exam; the parts are worth equal credit. The use of notes or other resources is not permitted. All work is to be your own.

Part A. Select four (4) of the following six problems to write for credit.

A1. State and prove the Intermediate Value Theorem.

A2. State and prove Urysohn's Lemma for metric spaces.

A3. Let X be a topological space and (Y, d) a metric space, consider the set $\mathcal{C}$ consisting of continuous functions from X to Y. Define the following topologies on the set $\mathcal{C}$: the topology of point wise convergence, uniform convergence, and compact convergence. Which topologies are contained in the others? Identify conditions on X and/or Y for when these topologies agree.

A4. Let $\phi : M \to N$ be a smooth map of manifolds. Define the concept of ϕ being an *immersion* and of ϕ being a *submersion*. Give an example of each concept in the case that not both M and N are Euclidean space.

A5. Complete the definition. A homology theory is a functor...

A6. For M a smooth manifold, define its de Rham complex $\Omega^*(M)$. State the Poincaré Lemma.

Date: August 2017.

Part B. Select four (4) of the following six problems to write for credit.

B1. Let X and Y be topological spaces, with Y being Hausdorff. Show that, for $f, g : X \to Y$ two continuous maps, the set

$$S \stackrel{\text{def}}{=} \{x \in X \mid f(x) = g(x)\}$$

is a closed subset of X.

B2. Prove that any $n \times n$ matrix with positive real entries has a positive real eigenvalue.

B3. You must solve both part (i) and part (ii) to receive full credit.

(i) Let D be a closed subset of a compact space X. Prove that D is compact.

(ii) Let C be a compact subset of a space Y. Give an example to show that C is not necessarily closed. What further assumptions on Y guarantee that C closed?

B4. Let $\ell, k \geq 1$ be integers. Consider the topological spaces $S^\ell \vee S^k$ and $S^\ell \times S^k$. Demonstrate a continuous embedding $S^\ell \vee S^k \to S^\ell \times S^k$. Prove, however, that there is no retraction of $S^\ell \times S^k$ onto $S^\ell \vee S^k$.

B5. Give an example of a topological space X such that X has the following cohomology groups:

$$H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = 1 \\ \mathbb{Z}/2 & k = 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z} & k = 3 \\ \mathbb{Z} & k = 4 \end{cases}$$

and further, there exists an interesting ring structure, i.e., not all cup products vanish. Prove that your example has the appropriate structure on its cohomology ring.

B6. Let $X = \mathbb{R}P^2 \times T^2$, further, fix $p_0 \in \mathbb{R}P^2$ and $t_0 \in T^2$. Compute $\pi_n(X, (p_0, t_0))$, for $n = 0, 1, 2, 3$.