Complex Analysis Ph. D. Comprehensive Exam

January 2018

Pick, circle, and solve 4 out of the following 5 problems.

1. Let z_0 be an isolated singularity of $f(z)$, and assume that $\lim_{z \to z_0} (z - z_0) f(z) = 0$. Show that z_0 is a removable singularity of $f(z)$.

2. Let $u : \mathbb{C} \to [0, \infty)$ be a non-negative harmonic function in the plane. Show that u is constant. (Hint: Start by arguing that u is the real part of an entire function f.)

3. Find a conformal map from the region $D = \{z \in \mathbb{C} : |z| < 1, |z - 1/2| > 1/2\}$ onto the unit disk $\mathbb{D}$.

4. Let f be a meromorphic function in the plane satisfying $f(z) = f(z+1) = f(z+i)$ for all $z \in \mathbb{C}$. Assume that f has m zeros and n poles (both counted with multiplicity) in the unit square $Q = [0,1]^2$, with none of these zeros or poles lying on the boundary ∂Q. Show that $m = n$. (Hint: Argument Principle)

5. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic mapping of the unit disk into itself with $f(0) = r \in (0, 1)$. Show that
$$\frac{r - |z|}{1 - r|z|} \le |f(z)| \le \frac{r + |z|}{1 + r|z|}$$
for all $z \in \mathbb{D}$.