PhD Real Analysis Comprehensive PhD Exam
(Jan 2018)

NAME:

Pick, Circle, and Solve 4 problems. Good luck!

1. Prove that $F(x) := \limsup_{n\to\infty} f_n(x)$ defines a measurable function of $x \in X$ if each $f_n : X \to \mathbb{R}$ is measurable (with respect to some σ-algebra $\mathcal{X} \subset 2^X$).

2. Given that $\lim_{n\to\infty} n^2\mu(E_n) = 0$, prove that μ-almost every x belongs to at most finitely many sets E_n. (Here $n \in \mathbb{N}$ and μ is a measure.)

3. Justify the convergence and compute the following limit (finite or infinite)

$$\lim_{n\to\infty} \int_1^\infty \frac{n\sin(x/n)}{x^p}\,dx \qquad (p \geq 2).$$

Note: You may want to treat differently the cases when $p > 2$ and when $p = 2$.

4. For $f \in L^1(\mathbb{R})$ and $h > 0$, prove the following equality of the L^1-norms: $\|f\|_1 = \|F\|_1$ where

$$F(x) := \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

5. Let $\mathcal{A} \subset 2^X$ be an algebra of sets and $\mathcal{X}$ be the σ-algebra generated by $\mathcal{A}$. Assuming that two finite measures μ and ν on $\mathcal{X}$ satisfy $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{A}$, show that $\mu(E) \leq \nu(E)$ for all $E \in \mathcal{X}$.

6. Let μ be a finite measure and $f \in L^\infty(X, \mathcal{X}, \mu)$. Show that the numbers $a_n := \|f\|_n^n = \int |f|^n\,d\mu$ satisfy

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \|f\|_\infty.$$