Applied Mathematics Ph. D. Comprehensive Exam

January 2018

Instructions: Answer 3 of the problems from **Part A** and 3 problems from **Part B**. Indicate clearly which questions you wish graded.

Part A

A.1 In the following we consider the system $Ax = b$ where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \quad , \quad b = \begin{pmatrix} \alpha \\ \alpha \\ 2 \end{pmatrix}$$

(a) Find all values of α for which $Ax = b$ has a solution.
(b) Find the least squares solution when $\alpha = 1$

A.2 Consider the perturbed eigenvalue problem

$$A(\varepsilon)x(\varepsilon) = \lambda(\varepsilon)x(\varepsilon) \quad , \quad A(\varepsilon) = \begin{pmatrix} 1-\varepsilon & 2 \\ 8 & 1+2\varepsilon \end{pmatrix}.$$

where $0 < \varepsilon \ll 1$ is a small parameter. Using the expansions

(1) $$A(\varepsilon) = A_0 + \varepsilon A_1 + \cdots$$
(2) $$x(\varepsilon) = x_0 + \varepsilon x_1 + \cdots$$
(3) $$\lambda(\varepsilon) = \lambda_0 + \varepsilon \lambda_1 + \cdots$$

and the Fredholm alternative, find λ_1 for the larger of the two eigenvalues of A.

A.3 Define the self adjoint operator $Lu \equiv u'' + u$ with domain

$$D(L) = \{u \in C^2[0,1] : u(0) = 0, u'(\pi) = 0\}$$

(a) Find the Greens function $g(x,t)$ for L.
(b) Use your result in (a) to verify the solution u of $Lu = f$ is given by

$$u(x) =< g, f >= \int_0^\pi g(x,t)f(t)dt$$

A.4 The Sturm Liouville operator $Lu = u''$ is defined on the domain:

$$D(L) = \{u \in C^2[0,\pi] : u'(0) = 0, u(\pi) = 0\}$$

(a) Find a complete set of eigenfunctions $\phi_n(x)$ for L.
(b) Use the eigenfunctions in (a) to find the series solution to

$$Lu = 1 \quad , \quad u \in D(L)$$

Part B

B.1 Define the functional

$$J(y) = \int_0^1 xy + (y')^2 \, dx$$

over the admissible set

$$A = \{y \in C^2[0,1] : y(0) = 0\}$$

a) Find the natural boundary condition at $x = 1$
b) Find the extrema of $J(y)$ over A

B.2 A geodesic on the cone $x^2+y^2 = z^2$ is parametrized in polar coordinates:

$$\begin{aligned} x &= r(\theta)\cos\theta \\ y &= r(\theta)\sin\theta \\ z &= r(\theta) \end{aligned}$$

for some function $r(\theta)$. Derive the functional

$$J(r) = \int_{\theta_1}^{\theta_2} L(r, r') \, d\theta$$

the geodesics extremize. Specifically, find L and then write out the Euler Lagrange equations associated with it (don't attempt to solve them).

B.3 Find two term expansions (in ϵ) for <u>all singular real</u> roots of

$$\epsilon x^4 - x - 1 = 0 \quad , \quad 0 < \epsilon \ll 1$$

B.4 Find the uniformly valid approximation for the boundary value problem:

$$\epsilon y'' + y' + xy^2 = 0 \quad , \quad y(0) = 0 \quad , \quad y(1) = 1$$

You may the only layer occurs at the boundary $x = 0$. Also note the leading outer problem is separable.