PhD Comprehensive Exam: ALGEBRA

August 2005

Instructions: Please put at most one problem per side of each sheet of paper turned in. Attempt all problems, showing all pertinent work.

(1) Suppose that $\mathbf{G}$ is an abelian group isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$ and having exactly 144 elements. Suppose $\mathbf{G}$ is generated by $\{x, y, z\}$ such that the following

relations hold: $\begin{array}{lcl} x^{15}\, y^3 & = & 1 \\ x^3\, y^7\, z^4 & = & 1 \\ x^{18}\, y^{14}\, z^8 & = & 1 \end{array}$.

How many elements of order 2 does G have?

(2) Suppose that $\mathbb{J}$ denotes the ideal in the polynomial ring $\mathbb{Z}[x]$ generated by 7 and $x - 3$. That is, $\mathbb{J} = \langle 7,\, x - 3 \rangle$. Prove that for each $p(x)$ in $\mathbb{Z}[x]$ there exists an integer a with $0 \le a \le 6$ such that $p(x) - a$ is in $\mathbb{J}$.

(3) Suppose that $\mathbb{F}$ is one of the fields $\mathbb{R}\,, \mathbb{C}\,, \mathbb{Q}$ or $\mathbb{Z}_5$. Suppose that the ideal $\mathbb{J}$ is generated by the polynomial $x^4 + 2x - 2$. For which choice of $\mathbb{F}$ is $\mathbb{F}[x]/\mathbb{J}$ a field?

(4) Suppose that $\alpha_1,\ \alpha_2$ and α_3 are the roots of $p(x) = x^3 + 7x^2 - 8x + 1$.

(a) Show that $p(x)$ is irreducible.

(b) Find the value of $\dfrac{1}{\alpha_1} + \dfrac{1}{\alpha_2} + \dfrac{1}{\alpha_3}$.

<u>SKETCH OF SOLUTIONS</u>

(1) We know that G is a product of cyclic groups from the fundamental theorem of abelian groups. The algorithm for finding the group is to row and column reduce the matrix of the defining relations, since row and column operations (without division) preserve a valid relation set:

$$\begin{pmatrix} 15 & 3 & 0 \\ 3 & 7 & 4 \\ 18 & 14 & 8 \end{pmatrix} \to \cdots \to \begin{pmatrix} 3 & -1 & 4 \\ 0 & 4 & -6 \\ 0 & 0 & 12 \end{pmatrix} \to \begin{pmatrix} 3 & -1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

This says there are elements a, b, c such that $b^4 = 1,\ c^{12} = 1$ and $a^3\, b^{-1} = 1$. But then $a^{12} = 1$ and $b = a^3$. It follows that a and c generators, each of order 12. So $m = n = 12$. G can be written as $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. There are three elements of order two.

(2) Since $x = 7 + (x-3)(2) \in \mathbb{J}$ it follows that only "constants" need be considered as representatives in the quotient space $\mathbb{Z}[x]/\mathbb{J}$. However, since multiples of 7 are in $\mathbb{J}$, the only possibility for $\mathbb{Z}[x]/\mathbb{J}$ is a quotient ring of $\mathbb{Z}_7$. There are no quotient rings other than $\mathbb{Z}_7$ or a trivial ring. So the set $\{0, 1, 2, 3, 4, 5, 6\}$ is a complete list of representatives.

(3) The answer depends only on whether $x^4 - 2x + 2$ is irreducible over the given fields, since maximal ideals correspond to irreducible generators. No polynomial of degree more than one is irreducible over $\mathbb{C}$. The polynomial is irreduible over $\mathbb{Q}$ by Eisenstein's criterion. The polynomial has the root $-1 \equiv 4$ in $\mathbb{Z}_5$. This leaves $\mathbb{R}$, but roots come in pairs of complex conjugates so this polynomial factors into (irreducible) quadratics over $\mathbb{R}$. So the answer is $\mathbb{Q}$.

(4) Let α_i be as in problem (4).
   (a) Reduce the polynomial mod 2 to get $x^3 + x^2 + 1$. If the cubic factors at all, it must have a linear factor, hence a root. No mod 2 numbers are roots, so this polynomial is irreducible.
   (b) First find a polynomial with conjugate roots $\dfrac{1}{\alpha_i}$, $i = 1, 2, 3$. Replace x by $1/t$ and multiply by t^3 to get $t^3 - 8t^2 + 7t + 1$ which must have roots $1/\alpha_i$. Examining the polynomial, the desired sum is given by the "trace" coefficient -8.