PhD Comprehensive Exam: ALGEBRA

August 2007

Name:

Instructions: Please put at most one problem per side of each sheet of paper turned in. Attempt all problems, showing all pertinent work.

1. Suppose that K is a normal subgroup of the finite group G, p is a prime divisor of $|K|$, and P is a p-Sylow subgroup of K. Prove that $G = KN_G(P)$, where

$$N_G(P) := \{g \in G \mid gPg^{-1} = P\}$$

is the normalizer of P in G.

2. Suppose that I and J are ideals in the commutative ring R such that $I + J = R$.

- **a:** Prove that $R/(I \cap J)$ and $R/I \times R/J$ are ring isomorphic.
- **b:** prove that if $f(x), g(x) \in \mathbb{Q}[x]$ are relatively prime, then for any $a(x), b(x) \in \mathbb{Q}[x]$ there is a $c(x) \in \mathbb{Q}[x]$ such that $f(x)|(a(x) - c(x))$ and $g(x)|(b(x) - c(x))$. To what extent is $c(x)$ unique?

3. Suppose that R is a commutative ring, M is an R-module, and N is a submodule of M.

- **a:** If M/N is free, prove that M is module isomorphic with $N \times (M/N)$.
- **b:** Give a counterexample to part a. if the assumption that M/N is free is dropped.

4. Let K be a (finite) Galois extension of $\mathbb{Q}$, let $G := gal(K/\mathbb{Q})$ and let $T : K \to K$ be defined by

$$T(a) := \sum_{\sigma \in G} \sigma(a).$$

- **a:** Prove that T is a vector space homomorphism (i.e., a linear map) and that $T(K) = \mathbb{Q}$.
- **b:** Suppose that $G = \langle \sigma \rangle$ is cyclic and let $\tau := \mathbf{1}_K - \sigma$, $\mathbf{1}_K : K \to K$ the identity automorphism. Prove that

$$\text{Image}(\tau) = \text{Ker}(T).$$