Ph.D. Comprehensive Examination: Applied Mathematics

August 19, 1994.

Instructions: In all of the following problems ε is a small parameter with $0 < \varepsilon << 1$. Attempt all questions.

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1. Consider the algebraic equation

$$\varepsilon u^3 + u^2 - f^2(x,y) = 0 \quad , \tag{1}$$

where $f(x,y)$ is a prescribed function on $(x,y) \in \mathbb{R}^2$.

a) Determine asymptotic expansions (with errors of $o(\varepsilon)$) for the three roots of (1).

b) Give an example of a function f for which the inner expansion in a) is no longer valid to $o(\varepsilon)$ for all (x,y).

2. Consider the following singularly perturbed boundary value problem:

$$\varepsilon u'' + (1+2x)u' + u = x \quad , u(0;\varepsilon) = 0 \quad , u(1;\varepsilon) = 1 \quad , \tag{2}$$

where $u = u(x;\varepsilon)$ and $(\ )'$ denotes differentiation with respect to x.

a) Show that (2) doesn't have a boundary layer at $x = 1$.

b) Determine an asymptotic solution $u_0(x;\varepsilon)$ of (2) with $u(x;\varepsilon) \sim u_0(x;\varepsilon) + O(\varepsilon)$ which is uniformly valid on $[0,1]$.

c) Define what is meant by a domain of common validity and how it relates to the answer in b).

d) How does the solution in b) change if the boundary conditions are changed to $u(0;\varepsilon) = -1/3\ ,\ u(1;\varepsilon) = 0$?

3. The Mathieu equation

$$y'' + (1 + 2\varepsilon \cos(2t))y = 0 \quad , y(0;\varepsilon) = a \quad , \quad y'(0;\varepsilon) = 0 \quad , (\ )' = d/dt(\ ) \quad , \tag{3}$$

models linear oscillators that are subject to frequency modulation.

a) Write (3) in standard form.

b) Solve the averaged equations.

c) Use the solution of b) to find an asymptotic solution $y_0(t;\varepsilon)$ of (3) which is valid to $O(1)$ for $t = O(1/\varepsilon)$.

d) Let $Y_0(\tau, \tilde{t})$ be a multiple scales solution of (3) which is asymptotic to y to $O(1)$ for $t = O(1/\varepsilon)$. Here the fast time τ satisfies the straining relation $d\tau/dt = \omega(\tilde{t})$ for some ω, $\tilde{t} = \varepsilon t$. Is it necessarily true that $y_0(t;\varepsilon) = Y_0(\tau, \tilde{t})$? Why or why not?

4. The Hodgkin-Huxley model for a space-clamped squid giant axon is given by the following set of equations:

$$C_m \frac{dV}{dt} = -\bar{g}_{Na} m^3 h (V - V_{Na}) - \bar{g}_K n^4 (V - V_K) - \bar{g}_L (V - V_L) + I_a \quad , \tag{4}$$

$$\frac{dm}{dt} = \frac{m_\infty(V) - m}{\tau_m(V)} \quad , \tag{5}$$

$$\frac{dh}{dt} = \frac{h_\infty(V) - h}{\tau_h(V)} \quad , \tag{6}$$

$$\frac{dn}{dt} = \frac{n_\infty(V) - n}{\tau_n(V)} \quad , \tag{7}$$

where V_{Na}, V_K and V_L are the Nerst potentials associated with sodium, potassium and leakage currents, respectively.

a) Write down the set of equations which correspond to pharmacologically blocking the potassium channels with TEA (tetraethylammonium).

b) Determine the applied current $I_a(t)$ (as a function of time) that is needed to voltage-clamp the axon at one half the sodium Nernst potential while the potassium channels are pharacologically blocked.

5. An analog to the Hodgkin-Huxley model described in problem 6 is the FitzHugh-Nagumo model

$$
\begin{aligned}
\frac{dv}{dt} &= f(v) - w + I_a \qquad (8)\\
\frac{dw}{dt} &= \varepsilon(v - w) \qquad (9)
\end{aligned}
$$

where $0 < \varepsilon << 1$, I_a is a constant and $f(v) = v(a - v)(v - 1)$. A more parsimonious analog model is obtained when f is replaced by the piecewise continuous function

$$
f(v) = \begin{cases} -v & v < 0 \\ v/2 & 0 \leq v \leq 2 \\ 3 - v & v > 2 \end{cases} \qquad (10)
$$

a) Determine an interval of values for I_a for which the model defined by (8)-(10) has a single unstable critical point and admits periodic solutions (to leading-order in ε).

b) Determine the leading-order period of the periodic solutions in a).