Ph.D. Comprehensive Examination: Applied Mathematics

January 11, 1999.

Instructions: Attempt all six questions.

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1. Compute two term asymptotic expansions for both roots of

$$\sqrt{2+x} - \varepsilon x^2 - 1 = 0 \quad , \quad 0 < \varepsilon \ll 1 \quad . \tag{1}$$

2. Consider the slowly varying oscillator problem for $y(t)$:

$$y'' + \omega^2(\tilde{t})y = 0 \quad , \quad \tilde{t} = \varepsilon t \quad , \quad 0 < \varepsilon \ll 1, \tag{2}$$

$$y(0) = a \quad , \quad y'(0) = 0 \quad , \tag{3}$$

and $(\ )' = \frac{d}{dt}(\ )$. Use a multiple scales technique with a strained fast time $\tau = \tau(\tilde{t})$ satisfying

$$\frac{d\tau}{dt} = \omega(\tilde{t}) \quad , \quad \tau(0) = 0 \tag{4}$$

and the expansion

$$y(t) = Y(\tau, \tilde{t}) \sim Y_0(\tau, \tilde{t}) + \varepsilon Y_1(\tau, \tilde{t}) + \cdots \quad , \tag{5}$$

to find a leading-order asymptotic solution of (2)-(3) valid for $t = O\left(\frac{1}{\varepsilon}\right)$ if $\omega(s) = 1 + s$.

3. Let Ω be the unit disk in $\mathbb{R}^2$ centered at the origin and $\partial\Omega$ its boundary. In polar coordinates (r, θ) the Laplacian of $u(r, \theta)$ is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad . \tag{6}$$

Use the method of matched asymptotic expansions to find a leading-order (radially symmetric) composite solution to the boundary value problem

$$\varepsilon \nabla^2 u - u = r \quad , \quad (r, \theta) \in \Omega \tag{7}$$

$$u = 0 \quad (r, \theta) \in \partial\Omega. \tag{8}$$

4. Let Ω be the unit disk in $\mathbb{R}^2$ centered at the origin and define the operator K on $L^2(\Omega)$ by

$$Ku = \int\int_{\Omega} k(r, r')u(r', \theta')dA \tag{9}$$

where the kernel, k, in polar coordinates is

$$k(r, r') = r(1 + r') \quad . \tag{10}$$

Find all nonzero eigenvalues λ and associated normalized eigenfunctions $\phi(r)$ of K.

5. Let

$$Lu \equiv \frac{d}{dx}\left(x\frac{du}{dx}\right). \tag{11}$$

Find a Green's function $g(x, t)$ satifying

$$Lg = \delta(x - t), \tag{12}$$

$$g(1, t) = 0, \tag{13}$$

$$g_x(2, t) = 0, \tag{14}$$

and then use it to find a Fredholm integral equation that all $C^2[1, 2]$ solutions $u(x)$ of

$$Lu + x = \lambda u \quad , \quad u(1) = 0 \quad , \quad u'(2) = 0 \tag{15}$$

must satisfy.

6. Solve the equation

$$u(x) - \lambda\int_0^{2\pi}\left(\cos x \cos t + \frac{1}{2}\cos 2x \cos 2t\right)u(t)dt = \frac{1}{2} + \frac{1}{2}\cos 2x \tag{16}$$

for the function $u(x)$, $0 \leq x \leq 2\pi$, and for any choice of the constant λ.