Ph.D. Comprehensive Examination:
Applied Mathematics

August 25, 1999.

Instructions: Answer all questions

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1. For what λ does the following integral equation have a solution? What is the solution for those λ?

$$u(x) + \lambda \int_0^{2\pi} \left(\sin x \sin t + \frac{1}{2} \sin 2x \sin 2t \right) u(t) dt = \sin(x) \tag{1}$$

2. Use a Green's function $g(x, \zeta)$ to solve

$$Lu \equiv \frac{d}{dx}\left(x \frac{du}{dx}\right) = f(x), \quad x \in (1,2), \tag{2}$$

$$u(1) = 1, \tag{3}$$

$$u'(2) = 1. \tag{4}$$

3. Let $D = \{\phi \in C^\infty(\mathbb{R}), supp(\phi) \;\; compact\}$.

a) Define what is meant by regular and singular distributions on D.

b) Is the following functional $\hat{t}$ a distribution? If so is it regular or singular? Explain why.

$$< \hat{t}, \phi >= \min_{x \in \mathbb{R}} \phi(x) \quad , \; \phi \in D \tag{5}$$

c) Show that $x\delta'(x) = -\delta(x)$ where $\delta(x)$ is the dirac delta function.

d) Use the result in c) to find any distributional solution $u(x)$ of

$$u''(x) = x\delta'(x). \tag{6}$$

4. Let Ω be some smooth domain in $\mathbb{R}^3$ and $\partial\Omega$ be its boundary. Define the functionals

$$J(u) = \int_\Omega |\nabla u(x)|^2 + u(x)^4 dx, \tag{7}$$

$$K(u) = \int_\Omega u(x)^2 dx, \tag{8}$$

and the admissible set

$$\mathcal{A} = \{u \in C^2(\bar{\Omega}) : K(u) = 1\}. \tag{9}$$

a) Derive the Euler-Lagrange equations and natural boundary conditions which a minimizer $\bar{u}(x)$ of

$$\min_{u \in \mathcal{A}} J(u) \tag{10}$$

must satisfy.

b) If the volume $vol(\Omega)$ of Ω is 2, find any upper bound for the minimum value of $J(u)$ over $u \in \mathcal{A}$.

5. In the following, I is the identity operator and the integral operator K is defined as

$$Ku \equiv \int_0^1 x^2 u(x) dx \tag{11}$$

on $L^2(0,1)$.

a) Define what is meant by the point spectrum $\sigma_p(K)$ and continuous spectrum $\sigma_c(K)$ of K.

b) Compute the point spectrum $\sigma_p(K)$.

c) Show that the resolvent operator $R_\lambda(K) \equiv (K - \lambda I)^{-1}$ is bounded when it exists.

d) State the spectral representation theorem for bounded operators on $L^2(0,1)$ and then use it to compute Lu when $u(x) = x$ and $L = exp(K)$.