Ph.D. Comprehensive Examination:
Applied Mathematics

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Instructions: Answer all questions

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1. Let $k : \mathbb{R}^2 \to \mathbb{R}$, $k(x, y) = x^2 y^2$ and define the operator

$$Ku \equiv \int_{-1}^{1} k(x, y) u(y) dy$$

on $L^2[-1, 1]$.

a) Let $\lambda_0 \neq 0$ be an eigenvalue of K and $u_0(x)$ be its associated normalized eigenfunction. Use the Fredholm alternative to determine λ_1 in the expansions

$$\begin{aligned} \lambda &= \lambda_0 + \epsilon \lambda_1 + O(\epsilon^2) \\ u(x, \epsilon) &= u_0(x) + \epsilon u_1(x) + O(\epsilon^2) \quad , \quad 0 < \epsilon \ll 1 \end{aligned}$$

where u is the eigenfunction of the perturbed eigenvalue problem:

$$Ku + \epsilon \int_{-1}^{1} x y^2 u(y) dy = \lambda u$$

b) Find a countable set of independent functions in the nullspace of K.

2. Let $p(x) > 0$ be smooth and define the Sturm-Liouville operator L with domain $D(L)$ as follows:

$$Lu \equiv \frac{d}{dx}\left(p(x)\frac{du}{dx}\right), \tag{1}$$

$$D(L) \equiv \{u \in L^2(0, \pi) : Lu \in L^2(0, \pi), u(0) = u(\pi) = 0\}. \tag{2}$$

a) By assuming that L has a complete orthonormal set of eigenfunctions $\phi_n(x), n = 1, 2, \ldots$ with associated eigenvalues $\lambda_n, n = 1, 2, \ldots$, show that the Green's function $g(x, y)$ solving $Lg = \delta(x - y)$ has the representation

$$g(x, y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}.$$

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b) For the case $p(x) = 1$, compute $g(x, y)$ and then use the result in a) to find a formula for the following sum:

$$\sum_{n=1}^{\infty} \frac{\sin nx \sin ny}{n^2}.$$

3. Let Ω be the unit sphere in $\mathbb{R}^3$ centered at the origin and define the functional

$$J(u) = \int_{\Omega} \| x \cdot \nabla u \|^2 \, dx \quad , x = (x_1, x_2, x_3).$$

Let $\bar{u}(x)$ minimize J over the admissible set

$$\mathcal{A} = \{u \in C^2(\Omega) : u|_{\partial\Omega} = 0, \| u \| = 1\}$$

where $\| \cdot \|$ is the norm in the $L^2(\Omega)$ sense. Derive an eigenvalue problem which $\bar{u}$ must satisfy by extremizing

$$H(u) = J(u) + \lambda \| u \|^2 .$$

4. The Fourier transform $\hat{u}(\lambda)$ of the scalar function $u(x)$ is defined by:

$$\hat{u}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{i\lambda x} dx.$$

a) Use the convolution theorem for Fourier transforms to find the function $u(x)$ which solves the scalar integral equation:

$$\int_{\mathbb{R}} e^{-|x-y|} u(y) dy + \alpha^2 u(x) = e^{-|x|} \quad , \quad \alpha \neq 0.$$

Note that, for all $a > 0$, if $f(x) = e^{-a|x|}$ then $\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+\lambda^2}$.

b) If $\alpha = 0$, what is the distributional solution of the integral equation?