Ph.D. Comprehensive Examination: Applied Mathematics
August 27, 2003.

Instructions: Answer four of the following six questions

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1. Let $k : \mathbb{R}^2 \to \mathbb{R}$, $k(x,y) = x^2y^2$ and define the operator

$$Ku \equiv \int_{-1}^{1} k(x,y)u(y)dy$$

on $L^2[-1,1]$. Let $\lambda_0 \neq 0$ be an eigenvalue of K and $u_0(x)$ be its associated normalized eigenfunction. Use the Fredholm alternative to determine λ_1 in the expansions

$$\begin{aligned} \lambda &= \lambda_0 + \epsilon\lambda_1 + O(\epsilon^2) \\ u(x,\epsilon) &= u_0(x) + \epsilon u_1(x) + O(\epsilon^2) \quad , \quad 0 < \epsilon \ll 1 \end{aligned}$$

where u is the eigenfunction of the perturbed eigenvalue problem:

$$Ku + \epsilon \int_{-1}^{1} xy^2 u(y)dy = \lambda u$$

2. Show that, as $x \to= \infty$,

$$\int_0^1 \frac{\sqrt{t(1-t)}}{(t+a)^x} dt \sim \left(\frac{a}{x}\right)^{3/2} a^{-x} \frac{\sqrt{\pi}}{2}$$

3. Let $p(x) > 0$ be smooth and define the Sturm-Liouville operator L with domain $D(L)$ as follows:

$$\begin{aligned} Lu &\equiv \frac{d}{dx}\left(p(x)\frac{du}{dx}\right), && (1) \\ D(L) &\equiv \{u \in L^2(0,\pi) : Lu \in L^2(0,\pi), u(0) = u(\pi) = 0\}. && (2) \end{aligned}$$

a) By assuming that L has a complete orthonormal set of eigenfunctions $\phi_n(x), n = 1,2,\ldots$ with associated eigenvalues $\lambda_n, n = 1,2,\ldots$, show that the Green's function $g(x,y)$ solving $Lg = \delta(x-y)$ has the representation

$$g(x,y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}.$$

b) For the case $p(x) = 1$, compute $g(x, y)$ and then use the result in a) to find a formula for the following sum:

$$\sum_{n=1}^{\infty} \frac{\sin nx \sin ny}{n^2}.$$

4. Let Ω be the unit sphere in $\mathbb{R}^3$ centered at the origin and define the functional

$$J(u) = \int_{\Omega} \| x \cdot \nabla u \|^2 \, dx \quad , x = (x_1, x_2, x_3).$$

Let $\bar{u}(x)$ minimize J over the admissible set

$$\mathcal{A} = \{u \in C^2(\Omega) : u|_{\partial\Omega} = 0, \| u \| = 1\}$$

where $\| \cdot \|$ is the norm in the $L^2(\Omega)$ sense. Derive an eigenvalue problem which $\bar{u}(x)$ must satisfy by extremizing

$$H(u) = J(u) + \lambda \| u \|^2 .$$

5. The Fourier transform $\hat{u}(\lambda)$ of the scalar function $u(x)$ is defined by:

$$\hat{u}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{i\lambda x} dx.$$

a) Use the convolution theorem for Fourier transforms to find the function $u(x)$ which solves the scalar integral equation:

$$\int_{\mathbb{R}} e^{-|x-y|} u(y) dy + \alpha^2 u(x) = e^{-|x|} \quad , \quad \alpha \neq 0.$$

Note that, for all $a > 0$, if $f(x) = e^{-a|x|}$ then $\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+\lambda^2}$.

b) If $\alpha = 0$, what is the distributional solution of the integral equation?

6. Consider a function $f(x)$ which is continuously differentiable except possibly at the set of points x_1, x_2, ..., x_n, at which points f has, possibly, finite jump discontinuities Δf_1, Δf_2, ..., Δf_n. Show that in the sense of distributions

$$f' = \frac{df}{dt} + \sum_{j=1}^{n} \Delta f_j \delta(x - x_j)$$

where df/dx is the usual derivative of f (wherever it exists).