Ph.D. Comprehensive Examination: Applied Mathematics
January 12, 2004.

Instructions: Attempt all four questions

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1. Find the $y(x) \in C^2[0,1]$ which minimizes the functional

$$J(y) = \int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right) dx$$

stating the natural boundary conditions y must satisfy.

2. Use a Green's function $g(x,y)$ to find the general solution of

$$\begin{aligned} Lu &\equiv \frac{d}{dx}\left(\frac{1}{x}\frac{du}{dx}\right) = f(x) \quad , \quad x \in (1,2) \\ u(1) &= 0 \\ u(2) &= 1 \end{aligned}$$

3. Let the integral operator K defined on $L^2(0,1)$ be given by

$$Ku \equiv \int_0^1 k(x,y)u(x)dx \; .$$

Further suppose that K is compact, self adjoint and has a complete set of mutually orthonormal eigenfunctions $\{\phi_n(x)\}$, $n = 1, 2, \ldots$ with associated eigenvalues $\{\lambda_n\}_{n\geq 1}$. Consider the following coupled integral equations:

$$u + Kw = f \qquad (1)$$

$$\beta w - Ku = g \qquad (2)$$

where $f, g \in L^2(0,1)$ and $\beta \in \mathbb{R}$.

a) Compute series solutions of $u(x)$ and $w(x)$ using $\{\phi_n(x)\}$ as a basis. For what β is such a solution valid?

b) When (1)-(2) has a nonunique solution, what conditions must $f(x)$ and $g(x)$ satisfy?

QUESTION FOUR ON BACK

4. The Fourier transform $\hat{u}(\lambda, t)$ of the scalar function $u(x, t)$ is defined by:

$$\hat{u}(\lambda, t) = \int_{\rm I\!R} u(x, t) e^{i\lambda x} dx$$

with associated inverse

$$\hat{u}(x, t) = \frac{1}{2\pi} \int_{\rm I\!R} \hat{u}(\lambda, t) e^{-i\lambda x} d\lambda$$

Now, consider the following initial value problem for $u(x, t)$

$$u_t = u_{xx} - tu \ , \quad x \in {\rm I\!R} \ , \ t > 0 \tag{3}$$

$$u(x, 0) = f(x) \tag{4}$$

where u is bounded.

a) Determine the Fourier transform $\hat{u}(\lambda, t)$ of the solution $u(x, t)$ of (3)-(4).

b) Use the Fourier convolution theorem and the identity

$$\frac{1}{2\pi} \int_{\rm I\!R} e^{-a\lambda^2/4} e^{-i\lambda x} d\lambda = \frac{1}{\sqrt{\pi a}} \, e^{-x^2/a} \quad , \quad a > 0$$

to determine an integral representation of the solution $u(x, t)$ of (3)-(4).