

APPLIED MATHEMATICS PRELIMINARY EXAM

Fall 2005

Instructions: Answer three problems from part A and three questions from part B. Indicate clearly which questions you wish to be graded.

Part A

A.1 (a) State in detail the Fredholm Theorem for

$$u(x) = f(x) + \lambda \int_a^b K(x, y)u(y)dy$$

(b) Solve the equation

$$\sin^2(s) = u(s) - \lambda \int_0^{2\pi} \sum_{k=1}^3 \frac{\cos(ks) \cos(kt)}{k} u(t)dt$$

for the function $u(s)$, $0 \leq s \leq 2\pi$, and for any choice of the constant λ .

A.2 (a) Determine conditions on f , α and β for which there are solutions of

$$\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1, \quad u(0) = \alpha, \quad u'(1) - u(1) = \beta.$$

(b) Find the “best least squares” solution of

$$\frac{d^2u}{dx^2} = 1, \quad u(0) = 1, \quad u'(1) - u(1) = 1.$$

A.3 (a) Suppose A is an $n \times m$ matrix. Define the unique pseudo inverse of A .

Do only one of parts b) or c) .

(b) Give a formula for the pseudo inverse of A which makes use of the LU decomposition of A . Prove that this is the pseudo inverse of A .

(c) Give a formula for the pseudo inverse of A which makes use of the singular value decomposition of A . Prove that this is the pseudo inverse of A .

A.4 Suppose that A is an $n \times n$ matrix with n distinct eigenvalues.

- (a) How are the eigenvalues of A and A^* related?
- (b) Show that the eigenvectors of A and A^* form a biorthogonal set, that is, of $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ are the eigenvectors of A and A^* , appropriately ordered, then $\langle \phi_j, \psi_k \rangle = 0$ if $k \neq j$.
- (c) Show that $\langle \phi_k, \psi_k \rangle \neq 0$.

A.5 Use a Green's function to solve

$$\frac{d}{dx}\left(x \frac{du}{dx}\right) = f(x), \quad x \in (1, 2), \quad u(1) = 1, \quad u'(2) = 1.$$

Part B:

B.1 Let f be a distribution.

- (a) Define what is meant by a weak solution of the equation

$$\Delta u = -f(x, y, z)$$

where Δ is the Laplacian in three dimensions.

- (b) Derive the solution of

$$\Delta u = -\delta(x, y, z) \text{ for } -\infty < x, y, z < \infty$$

Give a sketch of a proof that your solution is a weak solution to the equation.

B.2 Discuss the flow pattern around a circular obstacle associated with the complex potential

$$\Omega(z) = u_0\left(z + \frac{a^2}{z}\right) + \frac{i\gamma}{2\pi} \log z.$$

Show that $r = a$ is a streamline. Determine the asymptotic velocity when $z \rightarrow \infty$. Calculate the stagnation points. Sketch the flow when $\gamma = 0$.

B.3 Use Fourier transforms to show that the solution of the diffusion equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty$$

with initial data $u(x, 0) = h(x)$ takes the form

$$u(x, t) = \int_{-\infty}^{\infty} k(x - s, t) h(s) ds$$

where

$$k(x, t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}.$$

Determine $\lim_{t \rightarrow 0^+} k(x, t)$.

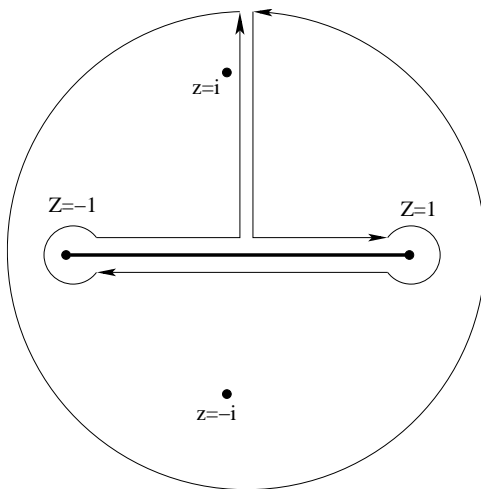
B.4 Consider the contour integral

$$J = \int_C \frac{(z^2 - 1)^{1/2}}{1 + z^2} dz$$

with the contour C shown below. By calculating the various contributions to J and using the residue theorem, show that

$$I := \int_{-1}^1 \frac{(1 - x^2)^{1/2}}{1 + x^2} dx = \pi(\sqrt{2} - 1)$$

The branch of the multivalued function $(z^2 - 1)^{1/2}$ should be fixed by choosing polar coordinates $z = 1 + \rho_1 e^{i\phi_1}$ and $z = -1 + \rho_2 e^{i\phi_2}$ with $0 \leq \phi_i \leq 2\pi$, $i = 1, 2$ so that the branch cut is the real axis between -1 and 1 .



B.5 Consider the variational problem

$$\int_a^b F(x, y(x), \frac{dy}{dx}(x)) dx = \min$$

for functions y and smooth F .

- (a) Derive the Euler equation for this problem.
- (b) Let $u(x, t)$ denote the deflection from rest of a uniform elastic string whose end points are held fixed. State Hamilton's principle for this problem. Use it to derive an equation for small deflections of the string from rest (ignore gravity).