

# APPLIED MATHEMATICS COMPREHENSIVE EXAM

Fall 2007

**Instructions:** Answer six of the following eight questions. Indicate clearly which questions you wish to be graded.

1. (a) Find the singular value decomposition,  $A = U\Sigma V^T$ , where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (1)$$

- (b) Find a the least squares solution of  $Ax = b$  with  $A$  given above and where

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (2)$$

- c. Find conditions on  $\vec{y}$  for which the system

$$A\vec{x} = \vec{y}$$

has a solution (A as above).

2. Consider the Boundary Value Problem

$$\frac{d^2u}{dx^2} + u = f(x), \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0.$$

where  $f$  is smooth.

- (a) Find the Greens function for the above problem.  
(b) Show that the solution to the boundary value problem depends continuously on the data  $f$ , using the sup norm for both  $f$  and  $u$ .

3. (a) State the Banach fixed point Theorem (also called the Contraction mapping principle).
- (b) Show that if the  $n$ th power of an operator, namely  $T^n$  for some  $n \geq 1$ , satisfies the condition of the Banach fixed point theorem, then the operator  $T$  itself has a unique fixed point.
- (c) Consider the initial value problem

$$x'(t) = f(x(t), t), x(0) = 1.$$

Assume that  $f$  is a continuous function satisfying the Lipschitz condition

$$|f(x, t) - f(y, t)| \leq M|x - y|$$

for some  $M$  and all  $x, y$  and  $t \in [0, T]$ .

Show that the initial value problem has a unique solution in  $C[0, T]$  by showing that the operator

$$T[x](t) = 1 + \int_0^t f(x(s), s)ds$$

has a fixed point.

4. Define the bounded linear operator  $T$  on  $L^2(0, 1)$  by

$$Tu(x) = \int_0^1 e^{-|x-y|}u(y)dy$$

- (a) Show that if  $Tu = v$  then  $v(x)$  satisfies

$$v'' - v = -2u, \quad 0 < x < 1 \text{ and } v(0) - v'(0) = v(1) + v'(1) = 0.$$

- (b) Show that if  $\lambda$  is a nonzero eigenvalue of  $T$ , with eigenfunction  $u(x)$  then

$$u'' + \left(\frac{2}{\lambda} - 1\right)u = 0, \quad 0 < x < 1 \text{ and } u(0) - u'(0) = u(1) + u'(1) = 0.$$

- (c) Show that the eigenvalues of  $T$  are real and lie in the interval  $(0, 2)$ .

5. (a) Define what is meant by a test function and a distribution.  
 (b) Let  $f(x) = |x|$  be defined on  $\mathbf{R}^1$ . Find the distributional derivatives  $f'$  and  $f''$  of  $f$  on  $\mathbf{R}^1$ . Show your work.
6. Consider the heat equation in radial coordinate

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \quad \text{for } r \in [0, a]$$

subject to boundary conditions

$$u|_{r=a} = 0, \quad t \geq 0,$$

and the initial condition

$$u|_{t=0} = f(r), \quad r \in [0, a].$$

- (a) Find the solution of the PDE by using separation of variables. Use the Bessel equation

$$J_0''(z) + \frac{1}{z} J_0'(z) + J_0(z) = 0, \quad Z > 0$$

and the fact that the Bessel function  $J_0$  has infinitely many zeros  $\{z_n\}_{n=1}^{\infty}$  on  $(0, \infty)$ .

- (b) Analyze the behavior of  $u(r, t)$  as  $t \rightarrow \infty$ .

7. Consider the eigenvalue problem:

$$y(x)'' + \lambda(1 + \epsilon x)y(x) = 0, \quad \text{on } [0, \pi],$$

$$y(0) = y(\pi) = 0.$$

Use a regular perturbation expansion to find approximate eigenvalues and eigenfunctions to order  $\epsilon$ .

8. Consider the variational problem:

Find the minimum of

$$J(x, y) := \int_0^\pi x'^2(t) + y'^2(t) + 2x(t)y(t)dt$$

for  $x, y \in C^2[0, \pi]$  satisfying  $x(0) = y(0) = x(\pi) = y(\pi) = 0$ .

(a) Derive the Euler equations for this problem.

(b) Does the functional reach a minimum?