Applied Mathematics Comprehensive Exam
January 2010

Instructions: Answer 3 of the problems from **Part A**, and answer 3 of the problems from **Part B**. Indicate clearly which questions you wish to be graded.

Part A

A.1 (a) Find the Singular Value Decomposition, $A = U\Sigma V^T$, where

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \\ -1 & 3 \end{bmatrix}$$

(b) Find conditions on $\vec{y}$ for which the system

$$A\vec{x} = \vec{y}$$

has a solution (where A is as given above).

(c) Find the least squares solution of $A\vec{x} = \vec{b}$ with A given above and

$$\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

A.2 (a) State in detail the Fredholm Alternative Theorem for

$$u(x) = f(x) + \lambda \int_a^b k(x,y)u(y)dy$$

(b) Solve the equation

$$u(x) = \sin^2 x + \lambda \int_0^{2\pi} \sum_{k=0}^{2} \cos(kx)\cos(ky)u(y)dy$$

for the function $u(x)$, $0 \le x \le 2\pi$, and for any choice of the constant λ.

A.3 Determine conditions on $f(x)$, α and β for which there are solutions of

$$u'' = f(x), \quad u(0) = \alpha, \;\; u'(1) - u(1) = \beta.$$

A.4 The sets of vectors $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ are said to be **biorthogonal** if $\langle \phi_i, \psi_j \rangle = \delta_{ij}$. Suppose that $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ are biorthogonal.

(a) Show that $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ each form a linearly independent set.

(b) Show that any vector in $\mathbb{R}^n$ can be written as a linear combination of $\{\phi_i\}_{i=1}^n$ as

$$x = \sum_{i=1}^{n} \alpha_i \phi_i$$

where $\alpha_i = \langle x, \psi_i \rangle$.

Part B

B.1 Define the bounded linear operator T on $L^2(0,1)$ by

$$Tu(x) = \int_0^1 e^{-|x-y|} u(y)\, dy$$

(a) Show that if $Tu = v$ then $v(x)$ satisfies

$$v'' - v = -2u, \quad 0 < x < 1 \quad \text{and} \quad v(0) - v'(0) = v(1) + v'(1) = 0$$

(b) Show that if λ is a nonzero eigenvalue of T, with eigenfunction $u(x)$ then

$$u'' + (\frac{2}{\lambda} - 1)u = 0, \quad 0 < x < 1 \quad \text{and} \quad u(0) - u'(0) = u(1) + u'(1) = 0$$

(c) Show that the eigenvalues of T are real and lie in the interval $(0,2)$.

B.2 Use the appropriate eigenfunction expansion to represent the solution of the given problem.

$$-u'' = f(x), \;\; 0 < x < \pi$$
$$u(0) = \alpha, \;\; u(\pi) = \beta$$

B.3 Use a Green's function to solve

$$u'' = f(x), \quad 0 < x < 1$$
$$u(0) = 1, \;\; u'(1) = 2.$$

B.4 Suppose that u_n is a sequence of generalized functions which are convergent in the sense of distributions.

(a) State the definition of the derivative, u_n', of the distribution u_n.

(b) Supposed that $u_n \to u$ in the sense of distributions. Show that $u_n' \to u'$ in the sense of distributions.

(c) Show that $\lim_{n\to\infty} \cos(nx) = 0$ in the sense of distributions.