Applied Mathematics Comprehensive Exam
August 2011

Instructions: Answer 3 of the problems from **Part A**, and answer 3 of the problems from **Part B**. Indicate clearly which questions you wish to be graded.

Part A

A.1 (a) Determine conditions on f, α and β for which there are solutions of

$$u'' = f(x), \quad 0 < x < \pi \qquad u'(0) = \alpha, \quad u'(\pi) = \beta$$

(b) Find the best possible (least squares) solution of

$$u'' = \sin^2(x), \quad u'(0) = 1, \quad u'(\pi) = 1$$

A.2 Let $A \in \mathbb{R}^{N \times N}$ be a Symmetric Positive Definite matrix. Recall the quadratic form $q(x) = \langle Ax, x \rangle$. Show that the level surface $q(x) = 1$ is an ellipsoid.

A.3 Use the Green's function to solve

$$u'' = f(x), \quad 0 < x < 1$$

$$u(0) = 3, \qquad \int_0^1 u(x)dx = 4.$$

A.4 Find solvability conditions for

$$u'' + u = f(x), \quad 0 < x < 2\pi$$

$$u(0) - u(2\pi) = \alpha, \;\; u'(0) - u'(2\pi) = -\beta$$

A.5 (a) Find the resolvent (or pseudo-resolvent) kernel for the problem

$$u(x) = f(x) + \lambda \int_0^1 u(y)dy$$

and for any choice of λ.

(b) Solve the integral equation for $f(x) = x$.

Part B

B.1 Define the bounded linear operator T on $L^2(0,1)$ by

$$Tu(x) = \int_0^1 e^{-|x-y|}u(y)\,dy$$

(a) Show that if $Tu = v$ then $v(x)$ satisfies

$$v'' - v = -2u, \quad 0 < x < 1 \quad \text{and} \quad v(0) - v'(0) = v(1) + v'(1) = 0$$

(b) Show that if λ is a nonzero eigenvalue of T, with eigenfunction $u(x)$ then

$$u'' + (\frac{2}{\lambda} - 1)u = 0, \quad 0 < x < 1 \quad \text{and} \quad u(0) - u'(0) = u(1) + u'(1) = 0$$

(c) Show that the eigenvalues of T are real and lie in the interval $(0,2)$.

B.2 Let $p(x) > 0$ be smooth and define the Sturm-Liouville operator L with domain $D(L)$ as follows:

$$Lu \equiv \frac{d}{dx}(p(x)\frac{du}{dx}),$$
$$D(L) \equiv \{u \in L^2(0,\pi)\colon Lu \in L^2(0,\pi),\ u(0) = u(\pi) = 0\}$$

(a) Assume that L has a complete orthonormal set of eigenfunctions $\phi_n(x), n = 1,2,\ldots$ with associated eigenvalues λ_n, for $n = 1,2,\ldots$. Show that the Green's function $g(x,y)$ solving $Lg = \delta(x-y)$ has the representation

$$g(x,y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}$$

(b) For the case $p(x) = 1$, compute $g(x,y)$ and then use the result in part a) to find a formula for the following sum:

$$\sum_{n=1}^{\infty} \frac{\sin nx \sin ny}{n^2}$$

B.3 Find the modified Green's function for

$$u'' = f(x), \quad 0 < x < 1, \qquad u(0) = u(1),\ u'(0) = u'(1)$$

B.4 Let $f(x)$ be a continuously differentiable function except at a discrete set of points $x_1, x_2 \ldots, x_n$, where f has finite jump discontinuities $\Delta f_1, \Delta f_2, \ldots, \Delta f_n$. Show that the derivative of $f(x)$ in the sense of distributions is given by

$$f' = \frac{df}{dx} + \sum_{j=1}^{n} \Delta f_j \delta(x - x_j),$$

where df/dx is the classical derivative of f wherever it exists.

B.5 Use Neumann iterates to solve the following integral equation using $u_0(x) = 1$.

$$u(x) = 1 + \int_0^x (y - x)u(y)dy$$

B.6 (a) Consider the variational problem of minimizing $J(y_1(t), y_2(t))$, where

$$J(y_1, y_2) = \int_{t_0}^{t_1} f(s, y_1(s), y_2(s), \dot{y}_1(s), \dot{y}_2(s))\, ds$$

subject to

$$y_1(t_0) = y_{1,0} \quad y_1(t_1) = y_{1,1}$$
$$y_2(t_0) = y_{2,0} \quad y_2(t_1) = y_{2,1}$$

Assume that $f(s, y_1, y_2, v_1, v_2)$ is a C^2 function. Derive the set of Euler-Lagrange equations for this problem.

(b) Find extremals of the functional

$$J(y, z) = \int_0^{\pi/2} [y']^2 + [z']^2 + 2yz\, ds$$

subject to $y(0) = 0, y(\pi/2) = 1, z(0) = 0, z(\pi/2) = 1$.