Complex Analysis

1. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is entire and $f(z) = f(1/z)$ for all $z \neq 0$. Prove that f is constant.

2. Given that $f\colon \{z\colon 1-\varepsilon < |z| < 1+\varepsilon\} \to \mathbb{C}$ is holomorphic for some $\varepsilon > 0$ and $\varphi : \mathbb{R} \to \mathbb{C}$ is defined by $\varphi(x) = f(e^{2\pi i x})$, prove that φ has a uniformly and absolutely convergent Fourier series.

3. If f is holomorphic on $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$, prove that

$$\left|\frac{f(z) - f(0)}{z}\right| \leq \left|1 - \overline{f(0)}f(z)\right| \quad \text{for all} \quad 0 < |z| < 1.$$

4. Suppose that Ω is a simply connected domain, $f\colon \Omega \to \mathbb{C}$ is holomorphic, and $f(z) \neq 0$ for $z \in \Omega$.

   (a) Prove that there is a holomorphic $g\colon \Omega \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ on Ω.

   (b) Assuming (a), prove that, given integers p and q, $q \neq 0$, there is a holomorphic $g\colon \Omega \to \mathbb{C}$ such that $(g(z))^q = (f(z))^p$ on Ω.

5. Let f be analytic in $|z| < 2 + \varepsilon$, for some $\varepsilon > 0$, except for a simple pole of residue $-R$ (negative R) at $z = 1$, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < 1$.

   Show that there is a constant B so that $|a_n - R| \leq B/2^n$ for $n = 0, 1, 2, \cdots$.

   [Hint: where is $f(z) + \frac{R}{z-1}$ analytic?]