Ph.D. Comprehensive Exam

Complex Analysis

August 27, 2001

1. Suppose that $f : \{z : |z| < 1\} \to \mathbb{C}$ is holomorphic and that $\sum_{k=0}^{\infty} f^{(k)}(o)$ converges. Prove that f extends to an entire function.

2. Are there nonconstant entire $f : \mathbb{C} \to \mathbb{C}$ with $f(z+1) = f(z) = f(z+i)$ for all $z \in \mathbb{C}$?

3. Suppose that P is a polynomial with $P(0) = 0$. Prove that for each $\delta > 0$ there exists an $\epsilon_0 > 0$ such that for $0 \le \epsilon < \epsilon_0$ $P(z) = \epsilon$ has a solutuion in $\{z : |z| < \delta\}$.

4. Suppose that $f : \{z : |z| \le 1\} \to \{z : |z| \le 1\}$ is continuous, holomorphic on $\{z : |z| < 1\}$, and that $f(0) = 0$. Suppose that T is an equilateral triangle with vertices on $|z| = 1$ and that $f(T) \subset T$. Prove that $f(z) = e^{i\theta} z$ for some $\theta \in \{0, 2\pi/3, 4\pi/3\}$.

5. Let S^1 be the unit circle $|z| = 1$ and suppose that $f : S^1 \to S^1$ is a nonconstant function that has a holomorphic extension to some open annulus A containing S^1. f is said to be *finite-to-one* if each point of S^1 has at most finitely many preimages, under f, on S^1 and f is said to be *bounded-to-one* if there is a $B < \infty$ such that each point of S^1 has at most B preimages, under f, on S^1.

   (a) Prove that f is finite-to-one.

   (b) Prove that f is bounded-to-one.