Complex Analysis Comprehensive Exam
August 22, 2007

Do all parts of the first problem and three of the remaining four.

1. True or false? Justify your answers.

(a) If $u : D \to \mathbb{R}$ is harmonic in a domain D, then $f = u_y + iu_x$ is analytic in D.

(b) If f is analytic for $|z| \leq 1$, with $f(0) = 1$, and $|f(z)| \geq 2$ for $|z| = 1$, then f has at least one zero in the unit disk.

(c) If f is analytic in $\mathbb{C} \setminus \{0\}$ and satisfies $f(2z) = f(z)$ for all $z \neq 0$, then f is constant.

2. Let $f(z) = \dfrac{z^2 - 1}{z^2 + 1}$. Determine and sketch the image under f of

(a) the unit disk $|z| < 1$,

(b) the sector $0 < \arg z < \pi/4$.

3. Let $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} \left(\frac{z}{1+n^2} \right)^n$.

(a) Show that this series defines f as an analytic function in $\mathbb{C} \setminus \{0\}$.

(b) Show that for every $w \in \mathbb{C}$ there exists a sequence (z_n) with $z_n \to 0$ and $\lim_{n\to\infty} f(z_n) = w$.

4. Assume that f is analytic near z_0 with $f'(z_0) \neq 0$. Show that

$$\int_{|z-z_0|=r} \frac{dz}{f(z) - f(z_0)} = \frac{2\pi i}{f'(z_0)}$$

for $r > 0$ sufficiently small.

5. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z = 1$ have in the unit disk? (Hint: Rouché's Theorem)