Complex Analysis Comprehensive Exam
August 25, 2008

Do all parts of the first problem and three of the remaining four.

1. True or false? Justify your answers.

(a) If f is analytic in $D = \mathbb{C} \setminus \{0\}$, then $g(z) = 1/\overline{f(1/\overline{z})}$ is analytic in D.

(b) If $u : \mathbb{C} \to \mathbb{R}$ is a non-constant harmonic function, then $\liminf\limits_{z\to\infty} u(z) = -\infty$ and $\limsup\limits_{z\to\infty} = +\infty$.

(c) If f is entire with $\lim\limits_{n\to\infty} \oint_{|z|=n} z^{-n} f(z)\, dz = 0$, then f is a polynomial.

(d) If f is a non-polynomial entire function, then $\liminf\limits_{z\to\infty} |f(z)| = 0$.

2. Determine the image of the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi/2\}$ under the mapping $f(z) = \dfrac{1}{1+e^z}$.

3. Let f be analytic near 0 and let $\alpha \in \mathbb{R}$ with $f(1/n) = n^\alpha$ for $n = 1, 2, 3, \ldots$.

(a) Show that α is a non-positive integer. (Hint: Showing $\alpha \leq 0$ is easy. Assuming that it is not an integer, consider the functions $z^{-m} f(z)$ for $m = 0, 1, 2, \ldots$ to arrive at a contradiction.)

(b) Show that f is a polynomial.

4. Let f be a meromorphic function in a domain D such that $\int_\gamma f(z)\, dz = 0$ for any closed curve γ in D which does not pass through a pole.

(a) Show that f can not have simple poles.

(b) Show that the assumptions do not imply that f is analytic, i.e., give an example where f has at least one pole.

5. How many solutions does the equation $z^5 = 2 + e^z$ have in the left halfplane? (Hint: Rouché's Theorem)