Complex Analysis PhD Comprehensive Exam

(Aug 2010)

Name:

If you rely on a theorem please state it carefully! *Good Luck!*

1. For $k > 0$, we denote by R_k the $2k$ by $2k$ square

$$R_k := \{z \in \mathbb{C} : \ |\mathrm{Im} z| < k, \ |\mathrm{Re} z| < k\}.$$

For f that is an analytic function defined on R_k for some $k > 1$ and taking values in the unit disk $D = \{|z| < 1\}$, show that

$$\sup_{z \in R_1} |f'(z)| \leq \frac{1}{(k-1)}.$$

(Note the smaller square R_1 in the sup.)

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic and $f(z+1) = f(z)$ for all $z \in G$ where G is some infinite subset of $\mathbb{C}$. Argue that $f(z+1) = f(z)$ for all $z \in \mathbb{C}$ if G is a bounded subset of $\mathbb{C}$ but it is not necessarily so if G is unbounded.

3. Consider f that is analytic on a domain containing the closure of the disk $D = \{|z| < 1\}$ and such that $f(D) \subset D$ and $f(\partial D) \subset \partial D$. Suppose there is $w_0 \in D$ that has only one preimage z_0 in D (i.e., $\{z_0\} = f^{-1}(w_0) \cap D$). Show that if $f'(z_0) \neq 0$ then f on D is of the form

$$f(z) = e^{i\theta} \frac{z-a}{1-\overline{a}z}$$

for some $a \in D$ and $\theta \in \mathbb{R}$. Give a counter-example with $f'(z_0) = 0$.

4. Find the annulus A of convergence for the Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} \left(\frac{2+e^k}{1+2e^k} \cdot z \right)^k$$

and explain why f has no anti-derivative on A (i.e., there is no analytic $F : A \to \mathbb{C}$ such that $F' = f$).

5. Evaluate the limit

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{\mathrm{Log}(1+z)}{z^2} dz$$

where $\epsilon > 0$ and C_ϵ is the quarter of the circle $\{|z| = \epsilon\}$ joining ϵ to $i\epsilon$. Justify your computation. (Here Log is the principal branch of the logarithm.)