Complex Analysis PhD Comprehensive Exam

(Aug 2010)

Name:

If you rely on a theorem please state it carefully! *Good Luck!*

1. For $k > 0$, we denote by R_k the $2k$ by $2k$ square

$$R_k := \{z \in \mathbb{C} : \ |\mathrm{Im} z| < k, \ |\mathrm{Re} z| < k\}.$$

For f that is an analytic function defined on R_k for some $k > 1$ and taking values in the unit disk $D = \{|z| < 1\}$, show that

$$\sup_{z \in R_1} |f'(z)| \leq \frac{1}{(k-1)}.$$

(Note the smaller square R_1 in the sup.)

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic and $f(z+1) = f(z)$ for all $z \in G$ where G is some infinite subset of $\mathbb{C}$. Argue that $f(z+1) = f(z)$ for all $z \in \mathbb{C}$ if G is a bounded subset of $\mathbb{C}$ but it is not necessarily so if G is unbounded.

3. Consider f that is analytic on a domain containing the closure of the disk $D = \{|z| < 1\}$ and such that $f(D) \subset D$ and $f(\partial D) \subset \partial D$. Suppose there is $w_0 \in D$ that has only one preimage z_0 in D (i.e., $\{z_0\} = f^{-1}(w_0) \cap D$). Show that if $f'(z_0) \neq 0$ then f on D is of the form

$$f(z) = e^{i\theta} \frac{z-a}{1-\overline{a}z}$$

for some $a \in D$ and $\theta \in \mathbb{R}$. Give a counter-example with $f'(z_0) = 0$.

4. Find the annulus A of convergence for the Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} \left(\frac{2+e^k}{1+2e^k} \cdot z \right)^k$$

and explain why f has no anti-derivative on A (i.e., there is no analytic $F : A \to \mathbb{C}$ such that $F' = f$).

5. Evaluate the limit

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{\mathrm{Log}(1+z)}{z^2} dz$$

where $\epsilon > 0$ and C_ϵ is the quarter of the circle $\{|z| = \epsilon\}$ joining ϵ to $i\epsilon$. Justify your computation. (Here Log is the principal branch of the logarithm.)

Gists of solutions:

1. For $z \in R_1$, the closed disk of radius $k-1$ around z fits inside R_k. Use the Cauchy formula for $f'(z)$ integrating along the boundary circle of this disk and estimate in the usual way (based on $|f| \leq 1$).

2. Bounded infinite G has an accumulation point, which is a non-isolated zero of $g(z) := f(z+1) - f(z)$. Hence g is the zero function.

3. There are several ways to argue here with varying degree of reliance on Schwarz Lemma, or its proof techniques, or its corollaries. In particular, invoking results about Blaschke products makes very quick work of it. Here is a less fancy line line of attack. By the argument principle, the number of preimages (in D) of $w_0 \in D$ is constant as w_0 varies over D. So this number must be one, making f an automorphism of D and forcing the form (by the standard characterization of automorphisms). This is of course if we do not neglect counting with multiplicities as evidenced by the example of $f(z) = z^2$ with $w_0 = 0$.

4. Use the "root test". That is, find the limits $\lim_{k\to\infty} \frac{2+e^k}{1+2e^k}$ and $\lim_{k\to-\infty} \frac{2+e^k}{1+2e^k}$ to get the radii of convergence of the power series $\sum_0^\infty$ (centered at 0) and $\sum_{-\infty}^0$ (centered at ∞) to see that $A = \{1/2 < |z| < 2\}$. Also, F does not exist because the coefficient at z^{-1} is nonzero making the integral along the closed loop $|z| = 1$ nonzero as well.

5. This can be viewed as a "fractional residue problem" but it is best attacked directly since the integral equals

$$\int_{C_\epsilon} \frac{1}{z} + [\text{ something analytic }] dz$$

where the $1/z$ part contributes a quarter of the usual $2\pi i$ (by a direct computation) and the contribution of the analytic part is vanishingly small as $\epsilon \to 0$ (if only because the length of C_ϵ tends to zero).