# Complex Analysis PhD Comprehensive Exam

*(Jan 2011)*

**Name:**

**Pick and <u>circle</u> four out of the five problems below, then solve them.**
**If you rely on a theorem please state it carefully!**
*Good Luck!*

1. For what real $\alpha, \beta > 0$ is there a single valued branch $f$ of the analytic function $z^\alpha(1-z)^\beta$ such that $f$ is defined on $\mathbb{C} \setminus [0,1]$? Justify your answer.

2. Let $E$ be the unit square $E = \{x + iy \in \mathbb{C} : \; x^2 \leq 1, \; y^2 \leq 1\}$ and $U : \mathbb{C} \setminus E \to \mathbb{C}$ be given by the following integral with respect to the ordinary area element $dxdy$

$$U(w) := \iint_E \frac{dxdy}{w - z} \qquad \text{where } z = x + iy \; .$$

Argue that $U$ is analytic at $w = \infty$, i.e., for $|w|$ large enough, $U(w)$ can be represented by a series in negative powers of $w$,

$$U(w) = b_0 + b_1 w^{-1} + b_2 w^{-2} + b_3 w^{-3} + \ldots,$$

and explicitly compute $b_1$ and $b_3$.

3. Suppose that a region $\Omega \subset \mathbb{C}$ has a boundary that is a simple smooth closed curve and let $\gamma : [0,1] \to \mathbb{C}$ be a smooth parameterization of that curve (with $\gamma(0) = \gamma(1)$). Let $f$ be a function that is analytic on a domain that contains the closure $\overline{\Omega}$ and such that

$$f \circ \gamma(t) = e^{i8\pi t}, \; t \in [0,1].$$

Prove that $f(\Omega) = \{z \in \mathbb{C} : \; |z| < 1\}$ and almost all $z$ with $|z| < 1$ have exactly four preimages in $\Omega$, i.e., $f^{-1}(z) \cap \Omega$ has four elements.

4. Let $f$ be analytic on the upper half-plane $H := \{x + iy : \; y > 0\}$ and such that $|f(z)| \leq 1$ for all $z \in H$ and $f(i) = 0$. Estimate as best as you can the modulus of the derivative $|f'(i)|$ and identify the $f$ for which the maximum modulus of $|f'(i)|$ is attained.

5. On the *infinite half-strip* $S \subset \mathbb{C}$ given by $S := \{x + iy : \; x > 0, \; y^2 < 1\}$ consider $f : S \to \mathbb{C}$ that is bounded, analytic, and extends continuously to the boundary $\partial S$. Show that, for all $z \in S$,

$$|f(z)| \leq \sup\{|f(w)| : \; w \in \partial S\}.$$

(Hint: First proceed under an additional assumption $\lim_{|z| \to \infty, z \in S} f(z) = 0$. Then consider $f_n(z) := f(z)e^{-z/n}$ to approximate $f(z)$.)