Complex Analysis PhD Comprehensive Exam

(Jan 2011)

Name:

Pick and <u>circle</u> four out of the five problems below, then solve them.
If you rely on a theorem please state it carefully!
Good Luck!

1. For what real $\alpha, \beta > 0$ is there a single valued branch f of the analytic function $z^\alpha (1-z)^\beta$ such that f is defined on $\mathbb{C} \setminus [0,1]$? Justify your answer.

2. Let E be the unit square $E = \{x+iy \in \mathbb{C} : \; x^2 \le 1, \; y^2 \le 1\}$ and $U : \mathbb{C} \setminus E \to \mathbb{C}$ be given by the following integral with respect to the ordinary area element $dxdy$

$$U(w) := \iint_E \frac{dxdy}{w-z} \qquad \text{where } z = x+iy \; .$$

Argue that U is analytic at $w = \infty$, i.e., for $|w|$ large enough, $U(w)$ can be represented by a series in negative powers of w,

$$U(w) = b_0 + b_1 w^{-1} + b_2 w^{-2} + b_3 w^{-3} + \ldots,$$

and explicitly compute b_1 and b_3.

3. Suppose that a region $\Omega \subset \mathbb{C}$ has a boundary that is a simple smooth closed curve and let $\gamma : [0,1] \to \mathbb{C}$ be a smooth parameterization of that curve (with $\gamma(0) = \gamma(1)$). Let f be a function that is analytic on a domain that contains the closure $\overline{\Omega}$ and such that

$$f \circ \gamma(t) = e^{i8\pi t}, \; t \in [0,1].$$

Prove that $f(\Omega) = \{z \in \mathbb{C} : \; |z| < 1\}$ and almost all z with $|z| < 1$ have exactly four preimages in Ω, i.e., $f^{-1}(z) \cap \Omega$ has four elements.

4. Let f be analytic on the upper half-plane $H := \{x+iy : \; y > 0\}$ and such that $|f(z)| \le 1$ for all $z \in H$ and $f(i) = 0$. Estimate as best as you can the modulus of the derivative $|f'(i)|$ and identify the f for which the maximum modulus of $|f'(i)|$ is attained.

5. On the *infinite half-strip* $S \subset \mathbb{C}$ given by $S := \{x+iy : \; x > 0, \; y^2 < 1\}$ consider $f : S \to \mathbb{C}$ that is bounded, analytic, and extends continuously to the boundary ∂S. Show that, for all $z \in S$,

$$|f(z)| \le \sup\{|f(w)| : \; w \in \partial S\}.$$

(Hint: First proceed under an additional assumption $\lim_{|z|\to\infty, z\in S} f(z) = 0$. Then consider $f_n(z) := f(z) e^{-z/n}$ to approximate $f(z)$.)

Hints for solutions.

1. There has to be no ambiguity of the complex argument of f upon circumventing the points 0 and 1, that is $2\pi\alpha + 2\pi\beta$ must be an integer multiple of 2π.

2. Expand the integrand into a power series in w^{-1} (which will converge uniformly on E). Interchange the summation and integration. Integrate to compute b_n. (Heck, get the formula for any b_n.)

3. Use the maximum principle to see that $f(\Omega) \subset D$, where $D := \{z : |z| < 1\}$. Invoke the argument principle (or the winding of $f \circ \gamma$ about z) to see that $z \in D$ has four preimages, if counted with multiplicity. (Conlude, in passing, that $D \subset f(\Omega)$, and so $f(\Omega) = D$.) If z is not a critical value, the true count gives the same.

4. Map H to the unit disk, and see that you are exactly in the context of the Schwarz Lemma.

5. Given $R > 0$, the maximum principle can be applied to $S_R := \{x + iy : R > x > 0,\ y^2 < 1\}$. Under the additional assumption, for $R > 0$ large, f is small on $S \setminus S_R$ and can be disregarded with little harm. For the general case, use the approximation, which happens to be uniform on any S_R.