Complex Analysis Ph. D. Comprehensive Exam
August 2012

Solve 4 of the following 5 problems.

1. Let $f(z)$ be analytic in a domain containing the closed unit disk $|z| \leq 1$. Show that

$$\int_{|z|=1} \overline{f(z)}\, dz = 2\pi i\, \overline{f'(0)}.$$

2. Let $D = \{z \in \mathbb{C} : |z| < 2 \text{ and } |z-1| < 1\}$.

(a) Sketch D.

(b) Find a conformal map from D to the upper halfplane.

3. Does there exist an analytic function f in the unit disk with

$$f\left(\frac{1}{n}\right) = \frac{1}{n^3} = f\left(-\frac{1}{n}\right)$$

for $n = 2, 3, \ldots$? Justify your answer.

4. Let D be a domain in the complex plane, let $f_1, \ldots, f_n : D \to \mathbb{C}$ be analytic functions, and let $g = |f_1| + \cdots + |f_n|$. Assume that g has a local maximum at a point $z_0 \in D$.

(a) Show that g is constant.

(b) Show that all the $f_1, \ldots, f_n$ are constant.

(Hint: Remember the proof of the maximum principle? Consider $G = e^{i\alpha_1} f_1 + \ldots + e^{i\alpha_n} f_n$ for suitably chosen $\alpha_k \in \mathbb{R}$.)

5. Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, with $a_n \neq 0$, be a polynomial of degree $n \geq 2$. The *fixed points* of f are the solutions of the equation $f(z) = z$. Assume that f has n distinct fixed points $z_1, \ldots, z_n$ with $\lambda_k = f'(z_k) \neq 1$ for $k = 1, \ldots, n$.

(a) By considering the integral $\displaystyle\int_{|z|=R} \frac{dz}{f(z) - z}$ for $R \to \infty$, derive an algebraic equation for that the λ_k have to satisfy.

(b) Assuming that $\lambda_1 = \ldots = \lambda_{n-1} = 0$, what is λ_n?

(c) Show that there exist indices k_1, k_2 with $\operatorname{Re} \lambda_{k_1} \geq 1$ and $\operatorname{Re} \lambda_{k_2} \leq 1$.