Complex Analysis PhD Comprehensive Exam

(Aug 2014)

Name:

Pick and <u>circle</u> four out of the five problems below, then solve them. If you rely on a theorem please state it carefully!
Good Luck!

1. Compute

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} \quad (a > 1).$$

2. Suppose that f is analytic on the half-plane $\{\mathrm{Re}(z) > 0\}$ and $|f(z)| \le 1$ there. Show that $|f(2)| \le 1/3$ if $f(1) = 0$.

3. On the annulus $\mathbb{A} := \{1/2 < |z| < 2\}$, consider a function f that is a limit of a normally convergent sequence $(p_n)_{n=1}^{\infty}$ of polynomials. Show that f extends analytically to the disk $\mathbb{D}_2 := \{|z| < 2\}$.

4. Suppose that f is analytic with $|f(z)| \le 1$ on the closed disk $\{|z| \le 1\}$. Show that if f vanishes at $a_1, a_2, a_3, \dots, a_N$ in the open disk $\{|z| < 1\}$, then

$$|f(0)| \le \prod_{k=1}^{N} |a_k|.$$

(Hint: See that $f_{\text{new}}(z) := f(z) \cdot \prod_{k=1}^{N} \frac{z - 1/\overline{a_k}}{1 - z/a_k}$ is also analytic with $|f_{\text{new}}(z)| \le 1$ on the closed disk $|z| \le 1$.)

5. Suppose that the series $f(z) = z + bz^2 + cz^3 + \dots$ gives an analytic function that maps a **bounded** open neighborhood U of $0 \in \mathbb{C}$ into itself, $f : U \to U$. Show that $b = 0$. (Hint: Denoting by $f_n : U \to U$ the n-fold composition $f_n := f \circ \dots \circ f$, find the formula for the coefficient b_n in the power series $f_n(z) = z + b_n z^2 + \dots$.)

Hints for solutions.

Ad 1. Vanilla via $\cos(\theta) = \frac{z+1/z}{2i}$ for $z = e^{i\theta}$. See page 203 in *Complex Analysis* by Gamelin (ISBN-10: 0-387-95069-9).

Ad 2. Reduce to Schwarz Lemma by precomposing f with a fractional linear transformation sending the unit disk onto the half-plane and mapping 0 to 1.

Ad 3. By the maximum principle, applied over the unit disk, the sequence of polynomials is Cauchy and thus uniformly convergent to some F. F is analytic on the unit disk and thus analytically extends f over the hole in the annulus.

Ad 4. Note that if some $a_k = 0$ then there is nothing to prove. Otherwise, use the hint as follows. The initial point is that the apparent singularity of $f_{\text{new}}(z)$ at $z = a_k$ (due to division by $1 - z/a_k$) is removable because f vanishes at a_k so that $f(z) = (z - a_k)\ldots = (1 - z/a_k)\ldots$. Thus f_{new} is analytic on the disk. On the boundary $|f_{\text{new}}(z)| \leq 1$. Invoke the maximum principle to finish.

Ad 5. Simply plugging the power series for f back into f repeatedly (and collecting the quadratic terms) yields $b_n = nb$. (Note that this uses the not-so-trivial theory of manipulation of power series.) Now, by the Cauchy formula for the second derivative, $b_n = f_n''(0)/2$ must be bounded because f_n are bounded (uniformly in n). This is only possible if $b = 0$.
Note: You can jazz this up to prove that $f(z) = z$ for all z.