Complex Analysis PhD Comprehensive Exam

(Jan 2012)

Name:

Pick and <u>circle</u> four out of the five problems below, then solve them. If you rely on a theorem please state it carefully!
Good Luck!

1. Show that the series

$$f(z) = \sum_{n=1}^{\infty} a_n \sin(nz)$$

defines an entire function if and only if the a_n converge to 0 super-exponentially, i.e., $\lim_{n\to\infty} a_n e^{n\alpha} = 0$ for all $\alpha > 0$. Assume $a_n > 0$ for simplicity.

2. Suppose that f is analytic at $z = 0$ and satisfies $f(z)^2 = z + f(z)$ for all z in a neighborhood of 0. Determine the radius of convergence of the power series expansion of $f(z)$ about $z = 0$. (<u>Note:</u> Do not compute the series.)

3. Consider the punctured upper half plane U and the annulus A as given by

$$U := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\} \setminus \{i\} \quad \text{and} \quad A := \{z \in \mathbb{C} : 1 < |z| < 2\}.$$

Show that there is an analytic $f : U \to A$ such that $f(U) = A$ and then briefly explain why such f cannot be one-to-one.
<u>Hint:</u> To construct f, you may use a branch of $F(z) = z^{ai}$ for a suitable $a > 0$.

4. Suppose $D := \{z \in \mathbb{C} : |z| < 1\}$ and $f : D \to D$ is analytic. Show that if there are two <u>distinct</u> $a, b \in D$ with $f(a) = a$ and $f(b) = b$ then $f(z) = z$ for all $z \in D$.

5. Assuming that $F : \mathbb{C} \to \mathbb{C}$ is analytic show that

$$\lim_{R\to\infty} \int_0^R F(x/R) e^{ix^2} dx = e^{\frac{i\pi}{4}} F(0) \int_0^{\infty} e^{-t^2} dt.$$

<u>Hint:</u> Integrate $F(z/R)e^{iz^2}$ over the triangle with vertices 0, R, and $R + iR$. Prove that the integrals along the horizontal side and the hypotenuse have the same limit as $R \to \infty$. Take the limit along the hypotenuse.