

**Ph.D COMPREHENSIVE EXAM: DYNAMICAL SYSTEMS**  
**1993**

**Instructions:** Please manage your time. Strive for completeness, but if you run short of time on any problem, indicate the method you would use to complete the problem.

**DO 5 OF THE FOLLOWING 7 PROBLEMS**

**Problem 1:** Let

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}$$

And let  $\phi_t$  be the flow on  $\mathbb{R}^4$  generated by  $x' = Ax$ .

- a) Solve for  $\phi_t$ .
- b) Show that  $\phi_t$  has an infinite number of invariant 2-tori for each  $\alpha$ .
- c) Find all  $\alpha \neq 0$  such that  $\phi_t$  has recurrent orbits which are not periodic or fixed.

**Problem 2:** Consider

$$\begin{aligned} x' &= \cos x, \\ y' &= y - y^3. \end{aligned}$$

- a) Sketch the phase portrait.
- b) Locate and classify three (3) non-colinear fixed points of the system above.
- c) Choose your favorite definition of “attracting set.” Then find a closed line segment in  $\mathbb{R}^2$  which is an attracting set for this system.

**Problem: 3** Let  $\Omega_\phi$  denote the nonwandering set of a  $C^1$  flow  $\phi_t$ .

- a) Show that  $\Omega_\phi$  is forward invariant.
- b) Show that  $\Omega_\phi$  is closed.
- c) Briefly describe a flow  $\phi_t$  such that  $\Omega_\phi$  has a dense orbit.
- d) Briefly describe a flow  $\phi_t$  such that  $\Omega_\phi$  has no dense orbit.

**Problem 4:** Give an example of a  $C^1$  diffeomorphism  $f : [0, 1] \rightarrow [0, 1]$  which is onto but not structurally stable in the  $C^1$  topology. Next, perturb  $f$  in the  $C^1$  topology to  $g$  in such a way that  $f$  and  $g$  are not conjugate.

**Problem 5:** Let  $\underline{a}, \underline{b} \in \{0, 1\}^{\mathbb{Z}}$  and let  $\sigma_2 : \{0, 1\}^{\mathbb{Z}} \rightarrow \{0, 1\}^{\mathbb{Z}}$  be the full 2-shift.

- a) Assume that  $\underline{a}$  and  $\underline{b}$  have different orbits under  $\sigma_2$ . Now give a condition on  $\underline{a}$  and  $\underline{b}$  which implies that their  $\omega$ -limit sets are identical. Show that your condition works.  
 b) Show the following: If  $\underline{a} = \sigma_2^m(\underline{b})$  for some  $m \in \mathbb{Z}$ , then

$$\mathcal{C}\uparrow\{\sigma_2^n(\underline{a}) : n \in \mathbb{Z}\} = \mathcal{C}\uparrow\{\sigma_2^n(\underline{b}) : n \in \mathbb{Z}\}.$$

(Here  $\mathcal{C}\uparrow$  denotes “closure”). Equivalently, show that if  $\underline{a}$  and  $\underline{b}$  have the same orbit under  $\sigma_2$ , then they have the same orbit closures.

- c) Find a weaker condition than that stated in part b) for two  $\sigma_2$  orbit closures to be equal. Show that your condition works.

**Problem 6:** Consider the planer system:

$$\begin{aligned} x' &= y \\ y' &= -ay + x - x^3 \quad \text{where } a \geq 0 \end{aligned} \tag{1}$$

- a) Find a first integral for (1) when  $a = 0$ . Use this conserved quantity to sketch the phase portrait for (1) at  $a = 0$ .  
 b) If  $a > 0$ , show that the omega-limit set  $\omega(x_0)$  for any  $x_0 \in \mathbb{R}^2$  is one of the equilibrium points. (Hint: use LaSalle’s Invariance Principle)  
 c) Let  $W^u(0, 0)$  denote the unstable manifold of the rest point  $(0, 0)$  and let

$$\mathcal{A} = W^u(0, 0) \cup \{(-1, 0)\} \cup \{(1, 0)\}.$$

Show that if  $a > 0$  then the  $\alpha$ -limit set  $\alpha(x_0)$ , for any  $x_0 \in \mathcal{A}$ , is an equilibrium point.

**Problem 7:** Consider the two dimensional system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

where  $f$  and  $g$  are  $C^1$  and are such that  $f(0, 0) = g(0, 0) = 0$ . Further let  $(0, 0)$  be the only critical point for the system. Assume that the disk  $D = \{(x, y) : x^2 + y^2 \leq 1\}$  is positively invariant.

- a) If  $(0, 0)$  is a source show, by use of a well-known theorem which you should state, that the system has a nontrivial periodic solution.  
 b) Now assume only that  $(0, 0)$  is unstable (i.e. not stable), what can your say about the existence of a nontrivial periodic solution?