Ph.D COMPREHENSIVE EXAM: DYNAMICAL SYSTEMS
1993

Instructions: Please manage your time. Strive for completeness, but if you run short of time on any problem, indicate the method you would use to complete the problem.

DO 5 OF THE FOLLOWING 7 PROBLEMS

Problem 1: Let

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}$$

And let ϕ_t be the flow on $\mathbb{R}^4$ generated by $x' = A\,x$.
a) Solve for ϕ_t.
b) Show that ϕ_t has an infinite number of invariant 2-tori for each α.
c) Find all $\alpha \neq 0$ such that ϕ_t has recurrent orbits which are not periodic or fixed.

Problem 2: Consider

$$\begin{aligned} x' &= \cos x, \\ y' &= y - y^3. \end{aligned}$$

a) Sketch the phase portrait.
b) Locate and classify three (3) non-colinear fixed points of the system above.
c) Choose your favorite definition of "attracting set." Then find a closed line segment in $\mathbb{R}^2$ which is an attracting set for this system.

Problem: 3 Let Ω_ϕ denote the nonwandering set of a C^1 flow ϕ_t.
a) Show that Ω_ϕ is forward invariant.
b) Show that Ω_ϕ is closed.
c) Briefly describe a flow ϕ_t such that Ω_ϕ has a dense orbit.
d) Briefly describe a flow ϕ_t such that Ω_ϕ has no dense orbit.

Problem 4: Give an example of a C^1 diffeomorphism $f : [0,1] \to [0,1]$ which is onto but not structurally stable in the C^1 topology. Next, perturb f in the C^1 topology to g in such a way that f and g are not conjugate.

Problem 5: Let $\underline{a}$, $\underline{b} \in \{0,1\}^{\mathbb{Z}}$ and let $\sigma_2 : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the full 2-shift.
a) Assume that $\underline{a}$ and $\underline{b}$ have different orbits under σ_2. Now give a condition on $\underline{a}$ and $\underline{b}$ which implies that their ω-limit sets are identical. Show that your condition works.
b) Show the following: If $\underline{a} = \sigma_2^m(\underline{b})$ for some $m \in \mathbb{Z}$, then

$$\mathcal{C}\updownarrow \{\sigma_2^n(\underline{a}) : n \in \mathbb{Z}\} = \mathcal{C}\updownarrow \{\sigma_2^n(\underline{b}) : n \in \mathbb{Z}\}.$$

(Here $\mathcal{C}\updownarrow$ denotes "closure"). Equivalently, show that if $\underline{a}$ and $\underline{b}$ have the same orbit under σ_2, then they have the same orbit closures.
c) Find a weaker condition than that stated in part b) for two σ_2 orbit closures to be equal. Show that your condition works.

Problem 6: Consider the planer system:

$$\begin{aligned} x' &= y \\ y' &= -ay + x - x^3 \qquad \text{where } a \geq 0 \end{aligned} \tag{1}$$

a) Find a first integral for (1) when $a = 0$. Use this conserved quantity to sketch the phase portrait for (1) at $a = 0$.
b) If $a > 0$, show that the omega-limit set $\omega(x_0)$ for any $x_0 \in \mathbb{R}^2$ is one of the equilibrium points. (Hint: use LaSalle's Invariance Principle)
c) Let $W^u(0,0)$ denote the unstable manifold of the rest point $(0,0)$ and let

$$\mathcal{A} = W^u(0,0) \cup \{(-1,0)\} \cup \{(1,0)\}.$$

Show that if $a > 0$ then the α-limit set $\alpha(x_0)$, for any $x_0 \in \mathcal{A}$, is an equilibrium point.

Problem 7: Consider the two dimensional system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

where f and g are C^1 and are such that $f(0,0) = g(0,0) = 0$. Further let $(0,0)$ be the only critical point for the system. Assume that the disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$ is positively invariant.
a) If $(0,0)$ is a source show, by use of a well-known theorem which you should state, that the system has a nontrivial periodic solution.
b) Now assume only that $(0,0)$ is unstable (i.e. not stable), what can your say about the existence of a nontrivial periodic solution?