Ph.D. Comprehensive Exam: Dynamical Systems 1997

Do Three of Problems 1-4.

1. Consider the 2-dimensional system:

$$\begin{aligned} x' &= ax - bxy - ex^2 \\ y' &= -cy + dxy - fy^2, \end{aligned} \tag{1}$$

where a, b, c, d, e, and f are all positive constants. Further, assume that $c/d > a/e$.

   a) Sketch the phase portrait for $x \geq 0$ and $y \geq 0$.
   b) Show that every solution $x(t)$, $y(t)$ of (1) with $x(0) > 0$, $y(0) > 0$ approaches the equilibrium solution $x = a/e$,$y = 0$.

2. Suppose that g is C^1 and b is a nonzero constant. Prove that $x'' + bx' + g(x) = 0$ has no non-equilibrium periodic solutions.

3. Let ϕ_t be a flow on $\mathbb{R}^n$ and $W \subset \mathbb{R}^n$ be a closed set. The immediate exit set of W is the set

$$W^- = \{x \in W : \phi_{[0,t]}(x) \not\subset W \text{ for all } t > 0\}.$$

The eventual exit set of W is the set

$$W^0 = \{x \in W : \phi_t(x) \notin W \text{for some } t > 0\}.$$

If $x \in W^0$, define the first exit time as

$$\tau(x) = \sup\{t \geq 0 : \phi_{[0,t)}(x) \subset W\}$$

Now consider the map $\Gamma : W^0 \to W^-$,

$$\Gamma(x) = \phi_{\tau(x)}(x)$$

Under the assumption that W^- is closed relative to W^0 prove that Γ is a continuous map.

4. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 and has Lipschitz constant L and suppose that $\phi_t(x_0)$ is a periodic solution of $x' = f(x)$ of period T. Find a lower bound for the period in terms of the Lipschitz constant. The optimal bound is $T \geq 2\pi/L$.

Do three of problems 5-8.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x/2, 2y - 7x^2)$. Prove that the parabola $\{(x, 4x^2) : x \in \mathbb{R}\}$ is the stable manifold of $(0, 0)$.

6. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and suppose that all periodic points of F are hyperbolic.

   a) Prove that F has at most countably may periodic orbits.

   b) Is each point of $\mathrm{Per}(F) = \cup_{k \geq 1} \mathrm{Fix}(F^k)$ isolated?

7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 5x + 4, & x \leq -2/5 \\ -5x, & -2/5 \leq x \leq 2/5 \\ 5x - 4, & x \geq 2/5 \end{cases}$$

   a) What is the Lyapunov exponent for any bounded orbit?

   b) How many points of least period 4 does f have?

   c) Prove that for all primes p, p divides $3^p - 3$. Hint: what is $\mathrm{Card}(\mathrm{Fix}(f^p))$?

8. Let $L(x) = \frac{1}{3}x$. Prove L is C^1- structurally stable on $\mathbb{R}$.