Dynamics Comprehensive Examination (Aug 2003)

Name:

Pick and circle five out of the seven following problems, then solve them.

If you rely on a theorem that goes beyond multivariable calculus, basic topology, or linear algebra please provide a formulation of the theorem. Good Luck!

1. Let X be a C^∞-smooth vector field on $\mathbb{R}^d$ and $\dot{x}(t) = X(x(t)), t \in [0,\infty), x(0) = x_0$ be the associated *Initial Value Problem (IVP)*. Consider the classical question: *Is there a global solution* $x : [0,\infty) \to \mathbb{R}^d$ *to (IVP) for all initial data* $x_0 \in \mathbb{R}^d$*?*
(a) Give an example showing the one cannot expect a positive answer without extra hypotheses. Prove that the example works.
(b) State (without proof) a condition on X (the more general the better) guaranteeing that the answer is "Yes".
(c) Prove that the answer is "Yes" for X with an additional property that, for some matrix A, $X(x+v) = X(x) + Av$ for all $x \in \mathbb{R}^d$ and $v \in \mathbb{Z}^d$.

2. Let A be a 2×2-matrix. In $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, consider the ODE given by

$$\dot{x} = Ax, \quad x \in \mathbb{R}^2$$

$$\dot{z} = -z^3 + |x|^2, \quad z \in \mathbb{R}.$$

(a) Linearize at 0 and show that 0 is never a hyperbolic stationary point.
(b) Show that if $\mathbb{R} \ni t \mapsto (x(t), z(t))$ is a solution and $\mathbb{R} \ni t \mapsto z(t)$ is periodic then so is $x(t)$.
Hint: You may want to verify that $\dot{x} = Ax$ has an orbit bounded away from 0 and ∞.
(c) Show that if all eigenvalues of A have negative real parts then 0 is a global attractor (i.e. any initial condition (x_0, z_0) evolves toward 0).

3. Let $f : X \to X$ be a homeomorphism of a compact metric space and let $N \geq 2$ be fixed. For each of the following assertions decide without proof if A implies B and if B implies A.
(a) A: f is topologically transitive; B: f^N is topologically transitive;
(b) A: f is topologically mixing; B: f^N is topologically mixing;
(c) A: f has an invariant measure; B: f^N has an invariant measure;
(d) Assuming additionally that $f(\Lambda) = \Lambda$ for a closed subset $\Lambda \subset X$:
A: Λ is a global attractor for f; B: Λ is a global attractor for f^N.

4. For $a > 1$, consider $f : [0,1] \to [0,1]$ given by

$$f(x) := \begin{cases} ax, & x \in [0, 1/a), \\ b(x - 1/a), & x \in [1/a, 1]. \end{cases} \tag{0.1}$$

where $b = (1 - 1/a)^{-1}$ so that $f(1) = 1$.
(a) Write out the explicit formula describing the action of the Perron-Frobenius-Ruelle operator P on $g : [0,1] \to \mathbb{R}$, and show that f preserves the Lebesgue measure $\lambda = dx$.
(b) Exhibit invariant measures for f other than λ. Can these be absolutely continuous with respect to λ?
(c) Prove that for any point $x \in [0,1]$, the preimage $f^{-n}(x)$ becomes dense in $[0,1]$ as $n \to \infty$ (i.e. for $\epsilon > 0$ any subsegment of length ϵ intersects $f^{-n}(x)$ for all sufficiently large n). Do you think that there is also $x \in [0,1]$ with $\{f^n(x) : n \in \mathbb{N}\}$ dense in $[0,1]$?

5. Below we are concerned with the unique fixed point x_A of a contraction $A : X \to X$ as provided by Banach Contraction Principle (BCP).
(a) State BCP, carefully.
(b) Explain how BCP is used to show existence and uniqueness of solutions to the initial value problem (IVP): $\dot{x} = F(x)$, $x(0) = x_0 \in \mathbb{R}^d$. Be brief, just give the form of A and X.
(c) One loosely says: *the unique fixed point x_A depends continuously on the contraction A*. Make this precise and give a proof.
(d) Explain in one sentence the relevance of (c) in the context of the IVP as discussed in (b).

6. Recall that, for a continuous map of a non-empty compact metric space $f : X \to X$, *the wandering set* consists of all points x for which some neighborhood U of x never comes back to itself: $f^n(U) \cap U = \emptyset$, $n > 0$. *The nonwandering set* $\Omega(f)$ is the complement of the wandering set.
(a) Show that $\Omega(f)$ is a non-empty closed set invariant under f.
(b) Show that if μ is a Borel measure invariant under f, then μ is supported on $\Omega(f)$ (i.e. $\Omega(f) = \{x : \mu(B_\epsilon(x)) > 0 \text{ for all } \epsilon > 0\}$).
(c) Give an explicit description (without proof) of the nonwandering set for the subshift of finite type over the alphabet $\{1, 2, 3, 4\}$ with the transition graph formed by the following edges: $1 \to 2$, $2 \to 1$, $2 \to 2$, $2 \to 3$, $3 \to 4$, $4 \to 3$.

7. Let A be a finite alphabet. Explain what it means for a subshift $X \subset A^{\mathbb{Z}}$ to be a subshift of finite type (SFT), and establish the following:
(a) intersection of two SFT's is an SFT;
(b) any subshift is a countable descending intersection of SFT's;
(c) there are subshifts that are not SFT.