Dynamics Comprehensive Examination (Aug 2004)

Name:

Pick and circle five out of the seven following problems, then solve them.

1. Consider the following system where $p, q \in \mathbb{R}^n$, $n \geq 1$.

$$\begin{cases} \dot{p} = -H_q - \nu p \\ \dot{q} = H_p \end{cases} \tag{1}$$

Above, $\nu > 0$ is a parameter, H is a C^∞-smooth function of p, q, and the subscripts signify partial differentiation, as usual. Let $\mathbb{R} \ni t \mapsto (p(t), q(t))$ be a solution of the initial value problem for the system. (Keep in mind that $p(t), q(t)$ depend on the initial condition and the parameter ν.)
(a) Assuming that the dot product $p \cdot H_p \geq 0$ for all p, q, show that H is non-increasing along the solution.
(b) Derive the system (the variational equations) for $\frac{\partial p}{\partial \nu}(t)$ and $\frac{\partial q}{\partial \nu}(t)$. What are the natural initial conditions for this system?
(c) Check that the flow of the system (1) must contract the volume in $\mathbb{R}^{2n}$.

2. (a) Show that topological entropy of a homeomorphism of the circle is zero.
(b) Give an example of a homeomorphism of the two-torus $\mathbb{T}^2$ with positive topological entropy; and identify that entropy (without giving a proof).
(c) Can topological entropy be infinite for an expansive homeomorphism of a compact metric space? Explain.

3. Let $T : X \to X$ be a measurable transformation preserving a probability measure μ. Below, A, B, C denote arbitrary finite measurable partitions of X, $H_\mu(A)$ is the entropy of A, and $H_\mu(B|C)$ is the conditional entropy of B with respect to C.
(a) In one sentence, explain the intuitive meaning of the following fundamental identity: $H_\mu(B \vee C) = H_\mu(C) + H_\mu(B|C)$.
(b) Use the identity to show that

$$\lim_{n\to\infty} \frac{1}{n} H_\mu(A \vee T^{-1}A \vee \cdots \vee T^{-n+1}A) = \lim_{n\to\infty} H_\mu(A|T^{-1}A \vee \cdots \vee T^{-n}A).$$

(c) For any increasing sequence $n_k \to \infty$ of natural numbers, show that

$$\lim_{k\to\infty} \frac{1}{k} H_\mu(A \vee T^{-n_1}A \vee \cdots \vee T^{-n_k}A) \geq h_\mu(T, A)$$

where $h_\mu(T, A)$ stands for the limit in (b).

4. Consider the Gauss map, that is $f : [0, 1] \to [0, 1]$ given by

$$f(x) := \begin{cases} \mathrm{frac}(1/x), & x \in (0, 1], \\ 0, & x = 0. \end{cases} \tag{2}$$

Here $\mathrm{frac}(x)$ is the fractional part of x.
(a) Is f expanding (i.e. $\inf_x |f'(x)| > 1$ where the inf is taken over all x for which $f'(x)$ is well defined)? How about its second iterate, f^2?
(b) Verify that the measure $\frac{1}{\log 2}\frac{dx}{1+x}$ is invariant under f.
(c) Do you think that f has other invariant measures that are absolutely continuous with respect to the Lebesgue measure? Explain.

5. Suppose that $A = (a_{ij})_{i,j=1}^d$ is a hyperbolic matrix (i.e. has no eigenvalues of unit modulus) with integer entries and $\det(A) = 1$. Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be the map induced by A on $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, $f :\ x(\mathrm{mod}\ \mathbb{Z}^d) \mapsto Ax(\mathrm{mod}\ \mathbb{Z}^d)$. Show the following.
(a) There is no $x_0 \in \mathbb{R}^d \setminus \{0\}$ with $\lim_{n\to\infty} A^n x_0 = 0$ and $\lim_{n\to-\infty} A^n x_0 = 0$; however, there exists $p_0 \in \mathbb{T}^d \setminus \{0\}$ with $\lim_{n\to\infty} f^n(p_0) = 0$ and $\lim_{n\to-\infty} f^n(p_0) = 0$.
(b) Assuming that p_0 is as in (a), the series $\sum_{n\in\mathbb{Z}} a_n f^n(p_0)$ converges for any binary sequence $(a_n)_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$.
(c) The mapping $h : (a_n) \mapsto \sum_{n\in\mathbb{Z}} a_n f^n(p_0)$ satisfies $h \circ \sigma = f^{-1} \circ h$ where σ is the shift on $\{0, 1\}^{\mathbb{Z}}$, i.e., $\sigma((a_n)) = (a_{n+1})$.

6. Fix an irrational $\alpha \in \mathbb{R}$. Consider $f : \mathbb{T}^2 \to \mathbb{T}^2$ given by $(x, y) \mapsto (x + \alpha, x + y)$ — all coordinates taken mod 1.
(a) Show that f is ergodic but not mixing or even weak mixing (with respect to the obvious Haar measure coming from $dxdy$).
(b) Find a closed formula for the n-th iterate, $f^n(x, y)$.
(c) Use (b) to show that the fractional parts of the sequence $n(n-1)\alpha/2$, $n \in \mathbb{N}$, are equidistributed in $[0, 1)$.

7. Let A and B be $d \times d$ matrices with non-negative integer entries and X_A and X_B be their edge shifts. (Recall, that X_A is essentially the set of all bi-infinite paths in the graph G_A with vertices $\{1, \ldots, d\}$ and a_{ij} edges from i to j.) Show the following.
(a) If X_A and X_B are conjugate (i.e. their shift dynamics are), then $\mathrm{trace}(A) = \mathrm{trace}(B)$.
(b) If $\mathrm{trace}(A^n) = \mathrm{trace}(B^n)$ for all $n \geq 0$, then $h_{\mathrm{top}}(X_A) = h_{\mathrm{top}}(X_B)$.
(c) Do you think that under the hypothesis of (b), X_A and X_B have to be conjugate? Explain.