

Ph.D. Comprehensive Exam In Dynamical Systems

august 23, 2010

- (1) Suppose that $f : X \rightarrow X$ is a homeomorphism of the compact metric space X and suppose that for some $x_0 \in X$ the ω -limit set of x_0 , $\omega(x_0)$, equals X . Let $G = \{x \in X : \omega(x) = X\}$. Prove that there are subsets U_1, U_2, \dots of X , each open and dense in X , with $\bigcap_{n \geq 1} U_n \subset G$.
- (2) Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^1 -diffeomorphism.
- a) If p is a periodic point of f of period m , prove that, for each $i \in \mathbb{Z}$, the eigenvalues of $Df^m(f^i(p))$ and $Df^m(p)$ are the same.
- b) Prove that if $p \in \text{Fix}(f) := \{x : f(x) = x\}$ is not isolated in $\text{Fix}(f)$, then 1 is an eigenvalue of $Df(p)$.
- (3) Let $f : S^1 \rightarrow S^1$ be a continuous degree 1 circle map and let μ be an ergodic invariant Borel measure for f . Prove that μ -almost every point has a well-defined rotation number under f and that this rotation number is constant on a set of full μ measure.
- (4) Let f be the hyperbolic toral automorphism defined by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Prove that:

- a) $\text{Per}(f) := \{x : x \text{ is periodic under } f\}$ is countable and dense in \mathbb{T}^2 .
- b) $\text{Hom}((0, 0)) := \{x : f^n(x) \rightarrow (0, 0) \text{ as } n \rightarrow \pm\infty\}$ is countable and dense in \mathbb{T}^2 .
- c) $\frac{\log(|\text{Fix}(f^n)|)}{n} \rightarrow \log\left(\frac{3+\sqrt{5}}{2}\right)$ as $n \rightarrow \infty$, where $|\text{Fix}(f^n)|$ denotes the cardinality of the set of points fixed by f^n .