Functional Analysis Ph.D. Comprehensive Exam

(Aug 2007)

Name:

Try to work out complete solutions to as many whole problems as you can.

If you rely on a theorem going beyond the basics of calculus, topology, or linear algebra please state it.

You may add reasonable assumptions if you cannot proceed without them. *Good Luck!*

1. Given a bounded linear operator on a complex Hilbert space, $A : H \to H$, the associated quadratic form $q : H \to \mathbb{C}$ is given by $q(x) := \langle Ax|x\rangle$. Recall that $\|q\| := \sup\{|q(x)| : \|x\| = 1\}$.
a) Which is always true: $\|A\| \leq \|q\|$ or $\|A\| \geq \|q\|$? (Justify and give a counter example where appropriate.)
b) Explain the idea of the proof of the implication $\|q\| = 0 \implies \|A\| = 0$.
c) Prove that $\|q\| = \|A\|$ provided $A = A^*$.

2. Consider the integral operator T on the Banach space $X := C([0,1])$ given by

$$Tx(t) := \int_0^t \sin(\pi s) x(s)\, ds.$$

a) Show that T is compact.
b) Prove that the spectrum of T equals $\{0\}$.
c) Write a series (using powers of T) for the solution x of the equation $x - Tx = y$ where $y \in X \setminus \{0\}$ is given.
d) What can you say about the rate of convergence of your series?

3. Let X_0 be the subspace of the space of all complex functions on $\mathbb{R}$ obtained as the (complex) linear span of the family of all *harmonics*: $\{t \mapsto e^{i\lambda t} : \lambda \in \mathbb{R}\}$.
a) Show that X_0 contains functions that are not periodic.
b) Give an argument for the existence of the following limit for any $x, y \in X_0$

$$\langle x|y\rangle := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\overline{y(t)}\, dt.$$

c) Prove that the Hilbert space X obtained as the completion of X_0 with respect to $\langle\cdot|\cdot\rangle$ (which is an inner product) is not separable?

4. a) Define the concept of a *closed linear operator* between two Banach spaces and give an example of an operator that is closed but not continuous.
b) Show that if A is closed and B is continuous, then $A + B$ is closed.
c) Demonstrate that any closed operator T can be represented as a composition $T = A \circ B^{-1}$ where A and B are continuous operators.

5. a) Write out C^∞ functions $k_n : [-1,1] \to \mathbb{R}$ such that, for every continuous $f : [-1,1] \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_{-1}^{1} k_n(t) f(t)\, dt = f(0).$$

b) Show that any functions k_n satisfying a) must necessarily satisfy $\sup_{n\in\mathbb{N}} \int_{-1}^{1} |k_n(t)|\, dt < \infty$.

6. Suppose that Y is a closed linear subspace of a Banach space X. Let $x \in X \setminus Y$ and $d > 0$ be the distance of x to Y. Consider a *minimizing sequence*, i.e. $(y_n) \subset Y$ with $\lim_{n\to\infty} \|x - y_n\| = d$.
a) Show that the sequence (y_n) may fail to converge (for some X, Y, x).
b) Show that (y_n) may even fail to have a convergent subsequence.
c) Show that (y_n) is guaranteed to converge if X is a Hilbert space?

7. Consider two linear normed spaces:

$$X := \{(x_k)_{k\in\mathbb{N}} : \lim_{k\to\infty} |x_k| = 0\} \quad \text{and} \quad Y := \{(x_k)_{k\in\mathbb{N}} : \exists_{C>0} \forall_{k\in\mathbb{N}}\ |x_k| < C/k\},$$

both taken with the *sup norm* (i.e. $\|(x_k)_{k\in\mathbb{N}}\| := \sup_{k\in\mathbb{N}} |x_k|$).
a) Show that X is complete (and thus a Banach space).
b) Show that Y is not complete.
c) Show that a continuous linear $k : X \to \mathbb{R}$ that vanishes on Y must be zero.

8. Fix $\alpha \in \mathbb{C} \setminus \{0\}$. Consider the operator L_α from $D_\alpha \subset L^2[0,1]$ to $L^2[0,1]$ where

$$D_\alpha := \{u : [0,1] \to \mathbb{C} : \ \ u \text{ is absolutely continuous, } u' \in L^2[0,1], \text{ and } u(1) = \alpha u(0)\}$$

and $L_\alpha u(t) := iu'(t)$ for a.e. $t \in [0,1]$.
a) Demonstrate that the domain of the adjoint L_α^* satisfies

$$D_\alpha^* \subset \{v : [0,1] \to \mathbb{C} : \ \ v \text{ is a.c., } v' \in L^2[0,1] \text{ and } v(1) = \overline{\alpha}^{-1} v(0)\}.$$

b) Carefully show that the above inclusion is, in fact, an equality.
c) For what values of the parameter α is L_α self-adjoint?

The gist of solutions:

1. a) The first fails for A that is a 2x2 nilpotent matrix. The second follows from Cauchy-Schwarz inequality.
b) Complex polarization allows to express $\langle Ax|y\rangle$ in terms of q (applied to $x \pm iy$).
c) The *real polarization* expresses $\langle Ax|y\rangle$ in terms of $q(x \pm y)$. By substituting βx for x and and $\beta^{-1}Ax$ for y where $\beta > 0$ is a parameter, we get $\langle Ax|Ax\rangle = \langle \beta Ax|\beta^{-1}Ax\rangle$ expressed in terms of q. Optimized the resulting bound over $\beta > 0$.

2. a) Apply the Arzela-Ascoli Theorem.
b) Estimate $\|T^n\| \leq C/n!$ by comparing with the vanilla antiderivative (via $|\sin(t)| \leq 1$).
c) The Neumann series.
d) The spectral radius is 0: $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$, so the convergence is faster than that of any geometric series.

3. a) vanilla. b) If $x(t) = \sum_{k=1}^{d} a_k e^{i\lambda_k t}$ and $y(t) = \sum_{k=1}^{d} b_k e^{i\lambda_k t}$, you have to use averaging of the function $f(x_1, \ldots, x_d) = (\sum_{k=1}^{d} a_k x_k)(\sum_{k=1}^{d} b_k x_k)$ over a suitable Kronecker flow on a d-dimensional torus $\mathbb{T}^d := \{(x_1, \ldots, x_d) : \ |x_i| = 1, \ x_i \in \mathbb{C}\}$.
c) Show that the harmonics are mutually orthogonal.

4. a, b) vanilla.
c) Recall the proof of CGT from OMT; A and B are the two projections from the graph of T.

5. a) Use (truncated) mollifiers.
b) Invoke a form of Uniform Boundedness Principle for the associated linear operators $K_n : C([0,1]) \mapsto \mathbb{R}$. The integrals are their norms.

6. a) Any norm with "squarish" balls will exhibit that; say $\mathbb{R}^2$ with the L^1 norm.
b) There is a standard example in $C([-1,1])$.
c) Hilbert spaces are sweet: by parallelogram inequality (y_n) must be Cauchy.

7. a) Vanilla. b,c) Just note that X is the closure of X.

8. Integrate by parts, gingerly, and with attention to the flow of logic!