Functional Analysis Ph.D. Comprehensive Exam

(Aug 2009)

Name:

Try to work out complete solutions to as many whole problems as you can.

If you rely on a theorem going beyond the basics of calculus, topology, or linear algebra please state it.

You may add reasonable assumptions if you cannot proceed without them. *Good Luck!*

1. For a normed linear space X and its dual space X', consider the two implications: (X' separable $\implies X$ separable) and (X separable $\implies X'$ separable).
a) Give a counter-example to the implication that is fale (and show that it works).
b) Give a proof of the implication that is true.

2. Let X be a real Banach space and $(x_n)_{n=1}^{\infty} \subset X$ be linearly dense ($\overline{\text{lin}\{x_n : n \in \mathbb{N}\}} = X$) and have the property that, for any square sumable $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$, we have

$$\left\| \sum_n a_n x_n \right\|^2 = \sum_n a_n^2.$$

a) Argue that X is a Hilbert space (i.e., its norm is induced by an inner product).
b) Assuming X is Hilbert, show that $(x_n)_{n=1}^{\infty} \subset X$ must be an orthonormal sequence in X.

3. a) Give an example of an unbounded linear functional $f : X \to \mathbb{R}$ on the space $X = l^2$ of square-summable sequences. (For partial credit, use a normed linear space X of your choosing.)
b) Argue that the kernel $N := \{x \in X : f(x) = 0\}$ cannot be closed for any f as in part a).

4. Consider a sequence $(A_n)_{n=1}^{\infty}$ in the space $L(X,Y)$ of all bounded linear maps between Banach spaces X and Y. Recall that (A_n) *converges weakly* iff there is $A \in L(X,Y)$ such that, for any $x \in X$ and $f \in Y'$, $\lim_{n\to\infty} f(A_n x) = f(Ax)$ and that (A_n) is *weakly Cauchy* iff $(f(A_n x))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ for any $x \in X$ and $f \in Y'$.
a) Argue that $\sup_n \|A_n\| < +\infty$ if (A_n) is weakly Cauchy.
b) State a general assumption on Y that guarantees that a weakly Cauchy (A_n) is actually weakly convergent.

5. Let X be a Banach algebra (with an identity). Recall that $x \in X$ is a *unit* iff x^{-1} exists in X and that $x \in X$ is a *topological divisor of zero* iff there is $(x_n) \subset X$ such that $\|x_n\| = 1$ yet either $x_n x \to 0$ or $x x_n \to 0$.
a) Show that if x is in the boundary of the set of all non-units, then x is a topological divisor of zero.
b) Describe explicitly the boundary of the set of all non-units for the algebra $C([0,1])$ of all continuous real valued functions on $[0,1]$.

6. On $X = L^2((0,\infty))$, consider the class of densely defined linear operators $A : X \supset D(A) \to X$ that are symmetric, i.e., $\langle Ax|y\rangle = \langle x|Ay\rangle$ for all $x, y \in D(A)$.
a) Argue that any A in this class with $D(A) = X$ must be bounded.
b) Show that $\sigma(A) \not\subset \mathbb{R}$ for $A = i\frac{d}{dt}$ defined on all smooth compactly supported $u : (0,\infty) \to \mathbb{C}$. (This A is in the class.)

7. On the space $L^2(\mathbb{Z})$ of two-sided square sumable sequences, consider the operator $A : (x_k)_{k=-\infty}^{\infty} \mapsto (x_{k-1} + x_{k+1})_{k=-\infty}^{\infty}$.
a) Explain why the residual spectrum of A must be empty.
b) Compute the spectrum $\sigma(A)$ and describe (as explicitly as you can) the spectral resolution $(E(\lambda))_{\lambda\in\mathbb{R}}$ associated to A.

8. Let $A : X \to X$ be a linear operator on a complex Hilbert space such that A^*A is compact.
a) Show that A is compact.
b) Show that $Y := \{x \in X : \; Ax + x = 0\}$ is finite-dimensional.
c) Assuming A is also self-adjoint and positive, explain how one can define $\sqrt{A}$.

The gist of solutions:

1. X' separable $\implies$ X separable, not vice versa because $(l^1)' \simeq l^\infty$ is not separable while l^1 is separable. The implication is a standard theorem the proof of which can be found in a text.

2. a) The hypothesis assures that X is isometric to l^2 via $(a_k) \mapsto \sum a_k x_k$. In particular, the norm on X satisfies the parallelogram identity.
b) Use the (real) polarization to compute $\langle x_k | x_l \rangle$ in terms of the norms $\|x_k \pm x_l\|$.

3. a) Extend the standard ON basis $(e_n)_{n\in\mathbb{N}}$ of l^2 to a basis $(e_n)_{\mathbb{N}\cup\Lambda}$ of l^2 as a linear space over $\mathbb{R}$. (This linear algebraic basis is often called a Hamel basis. Its existence rests on *Axiom of Choice* or *Transfinite Induction*. We note that the index set Λ is uncountable.) Take a nontrivial bounded linear functional, say $f_0((x_k)) := x_1$. Fix $n_0 \in \Lambda$. Let $f : l^2 \to \mathbb{R}$ be the unique linear functional given on the basis by $f(e_n) = f_0(e_n)$ for $n \neq n_0$ and $f(e_{n_0}) = f_0(e_{n_0}) + 1$. Then f is linear and everywhere defined and coincides with f_0 on the ON basis, so it would have to be equal to f_0 if f were continuous.
b) One quick argument is to note that if N is closed then f factors into the canonical projection $X \to X/N$ followed by a 1-dimensional linear map $X/N \simeq \mathbb{R} \to \mathbb{R}$, both of which are continuous. There is however a "more honest" direct argument ...

4. a) This is a standard fare resting on invoking PUB, twice.
b) Also standard. To define Ax, one needs reflexivity of Y in order to parley the bounded functional $Y' \ni f \mapsto \lim_{n\to\infty} f(A_n x)$ into an element of Y, which is the sought after Ax.

5. a) Suppose that x can be approximated by units x_n yet x^{-1} does not exist. Then it must be that $\|x_n^{-1}\| \to \infty$ (as otherwise Neumann series generates x^{-1} as a perturbation of x_n^{-1}). Hence

$$\frac{x_n^{-1}}{\|x_n^{-1}\|}x = \frac{x_n^{-1}}{\|x_n^{-1}\|}(x - x_n) + \frac{x_n^{-1}x_n}{\|x_n^{-1}\|} \to 0 + \frac{1}{\infty} = 0.$$

b) All continuous $f : [0,1] \to \mathbb{R}$ that have zeros but do not change sign.

6. a) This is a Herllinger-Toeplitz Theorem, which rests on checking that A is closed and applying CGT.
b) The equation $(A - \lambda I)u = f$ is an ODE that solves to

$$u(t) = e^{-i\lambda t}\left(u(0) + \frac{1}{i}\int_0^t e^{i\lambda s} f(s)ds\right).$$

Taking into account that $u(0) = u(t) = 0$ for $t > t_0$ where t_0 is sufficiently large (depending on u), we see that, for $f \in R(A - \lambda I)$,

$$\int_0^\infty e^{i\lambda s} f(s)ds = 0$$

which is to say that

$$\langle f|g_\lambda\rangle = 0$$

where $g_\lambda : s \mapsto e^{-i\overline{\lambda}s}$. Note that $g_\lambda \in L^2((0,\infty))$ as long as $Im(\lambda) > 0$, in which case g_λ belongs to $R(A-\lambda I)^\perp$ placing λ in the residual spectrum of A.

7. We all know that $2\pi = 1$.
a) The reason is that A is a bounded normal operator (so $\|Ax\| = \|A^*x\|$, which gives $R(A)^\perp = N(A^*) = N(A)$).
b) Use the Fourier series to move the dicussion to $L^2([0,1])$ (really to $L^2(\mathbb{T}^1)$) where A becomes the multiplication by $\phi : [0,1] \to \mathbb{R}$ given by $\phi(t) := 2\cos(t)$. Therefore, the spectrum equals the range of ϕ, which is $[-2,2]$. For $\lambda \geq -2$, the spectral resolution $E(\lambda)$ corresponds to the multiplication by the characteristic function $\chi_{\phi^{-1}([-2,\lambda])}$. For $\lambda \leq -2$, $E(\lambda) = 0$.

8. a) Use $\langle Ax_n - Ax_m|Ax_n - Ax_m\rangle = \langle A^*Ax_n - A^*Ax_m|x_n - x_m\rangle$ to see that if $(A_n^*Ax_n)$ is Cauchy then so is (Ax_n).
b) On Y, $A = -I$, which can only be compact if Y is fin-dim.
c) There are many ways. One is to write out the binomial power series (convergent for $0 \leq A \leq I$, which can be secured by scaling). Another is by squaring the eigenvalues and synthesizing $\sqrt{A}$ via the spectral resolution.