Numerical Analysis Preliminary Exam
Fall 2003

Do as many problems and parts of problems as time allows.

1. Consider the following problem

$$\begin{aligned} u_t &= (a(x)u_x)_x, \quad x \in (0,1), t > 0 \\ u(x,0) &= g(x), \quad x \in [0,1] \\ u(0,t) = 0, & \quad u(1,t) = 0, \quad t \geq 0, \end{aligned}$$

where $a(x) \geq c > 0$ for all $x \in [0,1]$. Applying the central difference operator to the problem above, we have the following SD scheme

$$v_l' = \frac{1}{(\Delta x)^2}[a_{l-1/2}v_{l-1} - (a_{l-1/2} + a_{l+1/2})v_l + a_{l+1/2}v_{l+1}], \quad l = 1, 2, \ldots d,$$

with $\Delta x = \frac{1}{d+1}$ and $a_k = a(k\Delta x)$, where k is an integer or a rational number.

   (a) Apply Euler's Method in order to derive a fully discrete (FD) scheme. Give the explicit FD scheme in matrix-vector form, and clearly identify the structure of the matrix and the vector.

   (b) Prove that the FD method is stable provided that $\mu \leq \frac{1}{2a_{\max}}$, where $a_{\max} = \max\{a(x) \colon 0 \leq x \leq 1\}$.

2. Consider the following PDE

$$\begin{aligned} u_t + au_x &= 0, \quad x \in (0,1), t > 0 \\ u(x,0) &= g(x), \quad x \in [0,1] \end{aligned}$$

and an appropriate boundary condition based on the sign of a. Assume that you are given an explicit FD scheme of the form

$$u_l^{n+1} = \sum_{k=-\alpha}^{\beta} c_k u_{l+k}^n$$

Assume that the coefficients of the FD scheme above are chosen so that the method has order of at least one. Argue that this sheme is only conditionally stable. That is, *There is no explicit (one-step) scheme of order at least one (consistent) for solving the hyperbolic PDE above that is unconditionally stable.*

3. Let $V = \{v \in C^2[0,1] \colon v(0) = \alpha\}$

$$f(u) = \int_0^1 \left[\frac{1}{2}p(x)(u')^2 + \frac{1}{2}q(x)(u)^2 - r(x)u\right] dx \quad \forall\, u \in V,$$

where $p \in C^1[0,1]$, $p(x) > 0$ for all $x \in [0,1]$, $q, r \in C[0,1]$ and $q(x) \geq 0$ for all $x \in [0,1]$. Derive the Euler-Lagrange D. E. Be sure to clearly identify the natural as well as the essential boundary conditions. Also clearly identify the linear space of test functions given by $\tilde{V}$.

4. Consider the following boundary value problem with $q(x) \geq 0 \, \forall \, x$

$$\begin{aligned} -u''(x) + q(x)u(x) &= f(x), \quad x \in (0,1), \\ u(0) = 0, & \quad u'(1) = 0, \end{aligned}$$

Give the variational formulation of the equation. Be sure to identify the function space, bilinear form and inner product that you are using.

5. Given the Lax-Friedrichs scheme,

$$u_l^{n+1} = \frac{1}{2}(1+\mu)u_{l-1}^n + \frac{1}{2}(1-\mu)u_{l+1}^n, \quad n \geq 0.$$

(a) Find the order of the method when applied to the advection equation

$$u_t + u_x = 0.$$

(b) Determine the range of Courant numbers μ for which the scheme is stable.

(c) Address the convergence properties of the scheme.

6. Consider the following boundary value problem

$$\begin{aligned} -u''(x) &= f(x), \quad x \in (0,1), \\ u(0) = 0, & \quad\quad u(1) = 0, \end{aligned}$$

Let S be a finite dimensional subspace of $H_0^1(0,1)$ which is spanned by the appropriate piecewise polynomial basis elements, and denote $u_S \in S$ as the finite element approximation to the true solution, u, of the boundary value problem above.

(a) Show that the approximation u_S exhibits an orthogonality property, that is

$$a(u - u_S, v) = 0 \quad \forall\, v \in S,$$

where

$$a(u,v) = \int_0^1 u'(x)v'(x)dx \quad \forall\, u, v \in H_0^1(0,1).$$

(b) Show that the approximation u_S is optimal in the energy norm; that is, show that

$$\|u - u_S\|_E = \min\{\|u - v\|_E : v \in S\}$$

7. Let $\|\cdot\|$ be a norm induced by an inner product $\langle \cdot, \cdot \rangle$ on a vector space V, and let S be a subspace of V. Fix $u \in V$.

(a) Show that if $\hat{u} \in S$ satisfies

$$\|\hat{u} - u\|^2 = \min_{s \in S} \|s - u\|^2,$$

then $\hat{u}$ satisfies the orthogonality condition

$$\langle \hat{u} - u, s \rangle = 0 \quad \text{for each } s \in S.$$

Here, we refer to $\hat{u}$ as the orthogonal projection of u onto the subspace S. HINT: For each $s \in S$, consider the function $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(t) = \|\hat{u} + ts - u\|^2$, and note that the function is minimized when $t = 0$.

(b) You may assume the result of part (a) above for this question. Let $\phi_1, \phi_2, \ldots \phi_N$ form a basis for S. Show that the orthogonal projection $\hat{u}$ of u onto S has a unique representation

$$\sum_{j=1}^{N} \alpha_j \phi_j,$$

where the vector $\alpha = [\alpha_1\ \alpha_2\ \ldots\ \alpha_N]^T$ satisfies $M\alpha = b$, with $b = [b_1\ b_2\ \ldots b_N]^T$

$$M_{ij} = \langle \phi_i, \phi_j \rangle, \quad b_i = \langle u, \phi_i \rangle, \quad 1 \leq i, j \leq N.$$