

Numerical Analysis Preliminary Exam

Fall 2005

Do as many problems and parts of problems as time allows.

1. Consider the following BVP

$$-\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) = f(x), \quad x \in (0, 1),$$
$$u(0) = 0, \quad u(1) = 0,$$

where $p \in C[0, 1]$ is positive and bounded away from zero and $f \in L^2(0, 1)$. Recall that the variational, or weak, formulation for this problem is to find $u \in V$ which satisfies

$$a(u, v) = F(v), \quad \text{for all } v \in V.$$

- (a) Specify the appropriate function space V , bilinear form $a(u, v)$ and linear functional $F(v)$ in the variational form.
- (b) Prove that the BVP has a weak solution and that this solution is unique.
- (c) Let S^h be a finite dimensional subspace of V with basis $\{\phi_1, \phi_2, \dots, \phi_n\}$. Give the linear system to be solved to obtain the Galerkin approximation, and prove that this system has a unique solution.
2. Show that the Crank-Nicolson method applied to the parabolic equation

$$u_t = \epsilon u_{xx}, \quad 0 < x < 1, \quad \epsilon > 0$$

$$u(0, t) = u(1, t) = 0, \quad t > 0$$

$$u(x, 0) = u^0(x), \quad 0 \leq x \leq 1$$

is unconditionally stable.

3. Consider the parabolic pde given by

$$u_t = u_{xx}, \quad x \in (0, 1), t > 0$$

$$u(x, 0) = g(x), \quad x \in [0, 1]$$

$$u(0, t) = u(1, t) = 0, \quad t \geq 0.$$

The Leapfrog Scheme is an explicit, 3-time-level scheme given by

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{(\Delta x)^2}$$

Show that for solutions which are sufficiently smooth, the truncation error is 2nd order in both time and space.

4. Consider the initial boundary value problem

$$\begin{aligned} u_t &= u_{xx}, & 0 < x < 1, \\ u(0, t) &= u(1, t) = 0, & t > 0 \\ u(x, 0) &= u^0(x), & 0 \leq x \leq 1. \end{aligned}$$

Consider the finite difference scheme given by

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} = \theta \frac{U_{i+1}^k - 2U_i^k + U_{i-1}^k}{(\Delta x)^2} + (1 - \theta) \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{(\Delta x)^2}$$

Determine the real values of the parameter θ for which the above difference scheme, obtained by solving for the U_i^{k+1} 's in terms of the U_i^k 's, is consistent, (conditionally) stable, and (conditionally) convergent.

5. Consider the Lax-Friedrichs scheme

$$U_j^{n+1} = \frac{1}{2}(1 + \nu)U_{j-1}^n + \frac{1}{2}(1 - \nu)U_{j+1}^n, \quad n \geq 0$$

applied to the one-way wave equation

$$u_t + au_x = 0,$$

with $\nu = \frac{a\Delta t}{\Delta x}$.

- Derive the CFL condition for this scheme. Be sure to clearly identify the pde domain of dependence as well as the numerical domain of dependence.
- Determine the range of ν for which the method is stable.
- Discuss the (conditional) convergence property of this scheme.

6. Given the functional $f : V \rightarrow \mathbb{R}$ of the form

$$f(u) = \int_a^b F(x, u, u') dx, \quad u \in V,$$

where $V = \{v \in C^2[a, b] : v(a) = 0, v(b) = 0\}$.

- Derive first-order directional derivative of f at the point u in the direction η , denote this variation by $f^{(1)}(u; \eta)$. Assume that the function F is sufficiently smooth for all the partial derivatives that you use to exist.
- Derive the Euler-Lagrange DE. Be sure to clearly identify the natural as well as the essential boundary conditions. Also clearly identify the linear space of test functions given by \tilde{V} .

7. Give the weak formulation of the following boundary value problem. Here the domain of definition $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega = \Gamma$, and assume $p(x, y) \geq p_0 > 0$.

$$\begin{aligned} -\nabla \cdot (p(x, y)\nabla u) &= f(x, y), & \forall (x, y) \in \Omega \\ u &= 0, & \forall (x, y) \in \Gamma. \end{aligned}$$