Numerical Analysis Preliminary Exam
Fall 2005

Do as many problems and parts of problems as time allows.

1. Consider the following BVP

$$
\begin{aligned}
-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) &= f(x), \quad x \in (0,1), \\
u(0) = 0, &\qquad u(1) = 0,
\end{aligned}
$$

where $p \in C[0,1]$ is positive and bounded away from zero and $f \in L^2(0,1)$. Recall that the variational, or weak, formulation for this problem is to find $u \in V$ which satisfies

$$
a(u,v) = F(v), \quad \text{for all } v \in V.
$$

(a) Specify the appropriate function space V, bilinear form $a(u,v)$ and linear functional $F(v)$ in the variational form.

(b) Prove that the BVP has a weak solution and that this solution is unique.

(c) Let S^h be a finite dimensional subspace of V with basis $\{\phi_1, \phi_2, \ldots \phi_n\}$. Give the linear system to be solved to obtain the Galerkin approximation, and prove that this system has a unique solution.

2. Show that the Crank-Nicolson method applied to the parabolic equation

$$
u_t = \epsilon u_{xx}, \quad 0 < x < 1, \;\; \epsilon > 0
$$

$$
u(0,t) = u(1,t) = 0, \quad t > 0
$$
$$
u(x,0) = u^0(x), \quad 0 \le x \le 1
$$

is unconditionally stable.

3. Consider the parabolic pde given by

$$
\begin{aligned}
u_t &= u_{xx}, \quad x \in (0,1), t > 0 \\
u(x,0) &= g(x), \quad x \in [0,1] \\
u(0,t) &= u(1,t) = 0, \quad t \ge 0.
\end{aligned}
$$

The Leapfrog Scheme is an explicit, 3-time-level scheme given by

$$
\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{(\Delta x)^2}
$$

Show that for solutions which are sufficiently smooth, the truncation error is 2nd order in both time and space.

4. Consider the initial boundary value problem

$$u_t = u_{xx}, \quad 0 < x < 1,$$

$$u(0,t) = u(1,t) = 0, \quad t > 0$$
$$u(x,0) = u^0(x), \quad 0 \le x \le 1.$$

Consider the finite difference scheme given by

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} = \theta \frac{U_{i+1}^k - 2U_i^k + U_{i-1}^k}{(\Delta x)^2} + (1-\theta)\frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{(\Delta x)^2}$$

Determine the real values of the parameter θ for which the above difference scheme, obtained by solving for the U_i^{k+1}'s in terms of the U_i^k's, is consistent, (conditionally) stable, and (conditionally) convergent.

5. Consider the Lax-Friedrichs scheme

$$U_j^{n+1} = \frac{1}{2}(1+\nu)U_{j-1}^n + \frac{1}{2}(1-\nu)U_{j+1}^n, \quad n \ge 0$$

applied to the one-way wave equation

$$u_t + au_x = 0,$$

with $\nu = \frac{a\Delta t}{\Delta x}$.

(a) Derive the CFL condition for this scheme. Be sure to clearly identify the pde domain of dependence as well as the numerical domain of dependence.

(b) Determine the range of ν for which the method is stable.

(c) Discuss the (conditional) convergence property of this scheme.

6. Given the functional $f : V \to \mathbb{R}$ of the form

$$f(u) = \int_a^b F(x, u, u')dx, \quad u \in V,$$

where $V = \{v \in C^2[a,b] : v(a) = 0, v(b) = 0\}$.

(a) Derive first-order directional derivative of f at the point u in the direction η, denote this variation by $f^{(1)}(u;\eta)$. Assume that the function F is sufficiently smooth for all the partial derivatives that you use to exist.

(b) Derive the Euler-Lagrange DE. Be sure to clearly identify the natural as well as the essential boundary conditions. Also clearly identify the linear space of test functions given by $\tilde{V}$.

7. Give the weak formulation of the following boundary value problem. Here the domain of definition $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega = \Gamma$, and assume $p(x,y) \ge p_0 > 0$.

$$-\nabla \cdot (p(x,y)\nabla u) = f(x,y), \quad \forall\ (x,y) \in \Omega$$

$$u = 0, \quad \forall\ (x,y)\ \in \Gamma.$$