NUMERICAL ANALYSIS PRELIMINARY EXAM

August 2006

1. Consider the ODE boundary value problem

$$-\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1,$$
$$u(0) + \frac{du}{dx}(0) = 0, \quad u(1) = 0.$$

Recall that the variational, or weak, formulation for this problem is to find $u \in V$ for which

$$a(u, v) = L(v), \quad \text{for all } v \in V.$$

a. Specify the appropriate vector space V and derive the functionals $a(u, v)$ and $L(v)$ in the variational form.

b. Verify that if u is a weak solution, and $u \in C^2[0, 1]$, then u solves the boundary value problem.

c. Explain how to implement the Ritz-Galerkin method for this problem.

2. Consider the parameterized family of time-marching methods

$$u_j^{n+1} - u_j^n = \frac{\Delta t}{h^2}[\alpha(u_{j-1}^n - 2u_j^n + u_{j+1}^n) + (1-\alpha)(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1})]$$

for the PDE $u_t = u_{xx}$, $0 < x < 1$, $t > 0$, with homogeneous Dirichlet boundary conditions, where $u_j^n \approx u(jh, n\Delta t)$, and α is real-valued. Determine the ranges of parameters α for which the time-marching method is unconditionally stable and for which it is always unstable.

For remaining values of α (where the method is conditionally stable) find conditions on h, Δt, and α which guarantee stability.

3. Consider the difference method

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \frac{u_{i-1,j}^n + u_{i+1,j}^n + u_{i,j-1}^n + u_{i,j+1}^n - 4u_{i,j}^n}{h^2}$$

for the 2-d heat equation,

$$u_t = u_{xx} + u_{yy}.$$

a. Perform a Von Neumann stability analysis to determine the range of values of Δt and h for which this method is stable.

b. Provide a general definition of local truncation error for evolution equations, and then compute the local truncation error for this method.

4. Derive the Gauss-Legendre quadrature formula,

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k=1}^{n} f(x_k)\,w_k,$$

which is exact for all polynomials of degree $\leq 2n-1$, i.e., explain how to obtain the x_k's and w_k's, and *prove* that your formula does indeed have the desired exactness. You may use the Hermite polynomial interpolation formula: If $f \in C^{2n}[a,b]$ and $\{x_k\}_{k=1}^{n}$ are distinct points in $[a,b]$, then

$$f(x) = \sum_{k=1}^{n} f(x_k)h_k(x) + \sum_{k=1}^{n} f'(x_k)\bar{h}_k(x) + \frac{f^{(2n)}(\xi)}{(2n-1)!}[\prod_{k=1}^{n}(x-x_k)]^2,$$

where for $k = 1, \ldots, n$,

$$\begin{aligned} h_k(x) &= [1 - 2(x-x_k)\ell_k'(x_k)][\ell_k(x)]^2, \\ \bar{h}_k(x) &= (x-x_k)[\ell_k(x)]^2, \\ \ell_k(x) &= \prod_{i=1, i\neq k}^{n} \frac{(x-x_i)}{(x_k-x_i)}. \end{aligned}$$

5. The leapfrog method for the 1-d linear avection equation

$$u_t + au_x = 0$$

is

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + a\frac{u_{i+1}^{n} - u_{i-1}^{n}}{2h} = 0.$$

Analyze the stability and accuracy of this method.