

# NUMERICAL ANALYSIS PRELIMINARY EXAM

August 2006

1. Consider the ODE boundary value problem

$$-\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1,$$
$$u(0) + \frac{du}{dx}(0) = 0, \quad u(1) = 0.$$

Recall that the variational, or weak, formulation for this problem is to find  $u \in V$  for which

$$a(u, v) = L(v), \quad \text{for all } v \in V.$$

- Specify the appropriate vector space  $V$  and derive the functionals  $a(u, v)$  and  $L(v)$  in the variational form.
  - Verify that if  $u$  is a weak solution, and  $u \in C^2[0, 1]$ , then  $u$  solves the boundary value problem.
  - Explain how to implement the Ritz-Galerkin method for this problem.
2. Consider the parameterized family of time-marching methods

$$u_j^{n+1} - u_j^n = \frac{\Delta t}{h^2} [\alpha(u_{j-1}^n - 2u_j^n + u_{j+1}^n) + (1 - \alpha)(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1})]$$

for the PDE  $u_t = u_{xx}$ ,  $0 < x < 1$ ,  $t > 0$ , with homogeneous Dirichlet boundary conditions, where  $u_j^n \approx u(jh, n\Delta t)$ , and  $\alpha$  is real-valued. Determine the ranges of parameters  $\alpha$  for which the time-marching method is unconditionally stable and for which it is always unstable.

For remaining values of  $\alpha$  (where the method is conditionally stable) find conditions on  $h$ ,  $\Delta t$ , and  $\alpha$  which guarantee stability.

3. Consider the difference method

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \frac{u_{i-1,j}^n + u_{i+1,j}^n + u_{i,j-1}^n + u_{i,j+1}^n - 4u_{i,j}^n}{h^2}$$

for the 2-d heat equation,

$$u_t = u_{xx} + u_{yy}.$$

- Perform a Von Neumann stability analysis to determine the range of values of  $\Delta t$  and  $h$  for which this method is stable.
- Provide a general definition of local truncation error for evolution equations, and then compute the local truncation error for this method.

4. Derive the Gauss-Legendre quadrature formula,

$$\int_{-1}^1 f(x) dx \approx \sum_{k=1}^n f(x_k) w_k,$$

which is exact for all polynomials of degree  $\leq 2n - 1$ , i.e., explain how to obtain the  $x_k$ 's and  $w_k$ 's, and *prove* that your formula does indeed have the desired exactness. You may use the Hermite polynomial interpolation formula: If  $f \in C^{2n}[a, b]$  and  $\{x_k\}_{k=1}^n$  are distinct points in  $[a, b]$ , then

$$f(x) = \sum_{k=1}^n f(x_k) h_k(x) + \sum_{k=1}^n f'(x_k) \bar{h}_k(x) + \frac{f^{(2n)}(\xi)}{(2n-1)!} \left[ \prod_{k=1}^n (x - x_k) \right]^2,$$

where for  $k = 1, \dots, n$ ,

$$\begin{aligned} h_k(x) &= [1 - 2(x - x_k) \ell'_k(x_k)] [\ell_k(x)]^2, \\ \bar{h}_k(x) &= (x - x_k) [\ell_k(x)]^2, \\ \ell_k(x) &= \prod_{i=1, i \neq k}^n \frac{(x - x_i)}{(x_k - x_i)}. \end{aligned}$$

5. The leapfrog method for the 1-d linear advection equation

$$u_t + au_x = 0$$

is

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + a \frac{u_{i+1}^n - u_{i-1}^n}{2h} = 0.$$

Analyze the stability and accuracy of this method.